Limits and Differentiation

1i. $\lim_{x\to\sqrt{3}} \dfrac{x^2-3}{x^2+3\sqrt{3}x-12}$ ii. $\lim_{x\to\sqrt{2}} \dfrac{x^4-4}{x^2+3\sqrt{2}x-8}$ iii. $\lim_{x\to 1} \dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$

iv. $\lim_{x\to 3}(x^2-9)\left[\dfrac{1}{x+3}+\dfrac{1}{x-3}\right]$ v. $\lim_{x\to 3}\left[\dfrac{1}{x-3}-\dfrac{3}{x(x^2-5x+6)}\right]$

vi. $\lim_{x\to 2}\left[\dfrac{1}{x-2}-\dfrac{4}{x^3-2x^2}\right]$ vii. $\lim_{x\to 2}\left[\dfrac{1}{x-2}-\dfrac{2(2x-3)}{x^3-3x^2+2x}\right]$

2).i. $\lim_{x\to 3}\dfrac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$ ii. $\lim_{x\to 0}\dfrac{\sqrt{2-x}-\sqrt{2+x}}{x}$ iii. $\lim_{x\to 1}\dfrac{\sqrt{1-x^2}+\sqrt{1-x}}{\sqrt{1-x^2}}$

iv. $\lim_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{2+3x}-\sqrt{2-3x}}$ v. $\lim_{x\to 1}\dfrac{\sqrt{1+2x}-\sqrt{3x}}{\sqrt{3+x}-2\sqrt{x}}$

3.i.If $\lim_{x\to a}\dfrac{x^5-a^5}{x-a}=405$, find all possible values of a *ii*. Evaluate $\lim_{x\to 2}\dfrac{(2x+4)^{\frac{1}{3}}-2}{x-2}$

iii.Find $\lim_{x\to -2}\dfrac{x^5+32}{x+2}$ iv. Find $\lim_{x\to 2}\dfrac{x^3-8}{x^2-4}$ v.Find $\lim_{x\to 3}\dfrac{x^5-243}{x^2-9}$

vii. Find $\lim_{x\to a}\dfrac{(x+2)^{\frac{5}{3}}-(a+2)^{\frac{5}{3}}}{x-a}$

4.i. $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$ ii. $\lim_{x\to\infty}\left(2x-\sqrt{4x^2+x}\right)$

iii. $\lim_{x\to\infty}\dfrac{4x^2+5x+6}{3x^2+4x+5}$ iv. $\lim_{x\to\infty}\left(\sqrt{x^2-8x}+x\right)$

5.i. $\lim_{x\to 0}\dfrac{1-\cos x}{x\sin x}$ ii. $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$ iii. $\lim_{x\to 0}\dfrac{x\cos x+\sin x}{x^2+\tan x}$

iv. $\lim_{x\to 0}\dfrac{1-\cos 2x+\tan^2 x}{x.\sin x}$ v. $\lim_{x\to 0}\dfrac{\sqrt{5}-\sqrt{4+\cos x}}{x.\sin x}$

6.i. $\lim_{x\to 0}\dfrac{\tan x-\sin x}{1-\cos x}$ ii. $\lim_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}$ iii. $\lim_{x\to 0}\dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$

iv. $\lim_{n\to\infty} 2^{n-1}.\sin\left(\dfrac{a}{2^n}\right)$ v. $\lim_{n\to\infty}\dfrac{\sin\left(\dfrac{a}{2^n}\right)}{\sin\left(\dfrac{b}{2^n}\right)}$

vi. $\lim_{x \to 0} \dfrac{1-\cos(1-\cos x)}{x^4}$ vii. $\lim_{x \to \infty} n \sin\left(\dfrac{\pi}{4n}\right).\cos\left(\dfrac{\pi}{4n}\right)$

7i. $\lim_{x \to 1} \dfrac{1-x}{(\cos^{-1} x)^2}$ ii. $\lim_{x \to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1} x}$ iii. $\lim_{x \to 0} \dfrac{1}{x} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

iv. $\lim_{x \to 0} \dfrac{x(1-\sqrt{1-x^2})}{\sqrt{1-x^2}\,(\sin^{-1} x)^3}$ v. $\lim_{x \to \frac{1}{\sqrt{2}}} \dfrac{x-\cos(\sin^{-1} x)}{1-\tan(\sin^{-1} x)}$ vi. $\lim_{x \to 1} \dfrac{1-x}{\pi - 2\sin^{-1} x}$

vii. $\lim_{x \to 0} \dfrac{\sin^{-1} x - 2x}{\sin^{-1} x - 2\sin\left[\frac{1}{2}\sin^{-1} x\right]\left[3-4\sin^2\left(\frac{1}{2}\sin^{-1} x\right)\right]}$ viii. $\lim_{x \to 0} \dfrac{1}{x} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

8.i. $\dfrac{e^{2+x}-e^2}{x}$ ii. $\lim_{x \to \frac{\pi}{2}} \dfrac{e^{\cos x}-1}{\cos x}$ iii. $\lim_{x \to 0} \dfrac{e^{\tan x}-1}{x}$

iv. $\lim_{x \to 0} \dfrac{e^x - e^{\sin x}}{x-\sin x}$ v. $\lim_{x \to 0} \dfrac{x(e^{2+x}-e^2)}{1-\cos x}$ vi. $\lim_{x \to 0} \dfrac{e^{ax}-1}{\sin x}$

9.*a*.Test the continuity of the function

$$f(x) = \begin{cases} (x-a).\sin\dfrac{1}{(x-a)}; & x \neq a \\ 0 & x = a \end{cases}$$ at $x = a$.

b.Test the continuity of the following functions at $x = 0$

i. $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}; & x \neq 0 \\ 0 \quad ; & x = 0 \end{cases}$ ii. $f(x) = \begin{cases} \dfrac{x(3e^{\frac{1}{x}}+4)}{2-e^{\frac{1}{x}}}; & x \neq 0 \\ 0 \quad ; & x = 0 \end{cases}$

iii. $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}+e^{\frac{-1}{x}}}{e^{\frac{1}{x}}-e^{\frac{-1}{x}}}; & x \neq 0 \\ 0 \quad ; & x = 0 \end{cases}$

10 Determine the values of *a,b,c* for which the function

$$f(x)\begin{cases} \dfrac{\sin(a+1)x+\sin x}{x}; & x < 0 \\ c \quad ; & x = 0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{b\sqrt{x^3}} \quad ; & x > 0 \end{cases}$$

may be continuous at $x = 0$

11.Let $f(x)\begin{cases} \dfrac{1-\cos 4x}{x^2} & ; \quad x<0 \\ a & ; \quad x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & ; \quad x>0 \end{cases}$

show for the function $f(x)$ to be continuous at $x=0$ that $a=8$.

12.Find the value of a and b which make the function

$$f(x)=\begin{cases} \dfrac{1}{3} & x<\dfrac{\pi}{2} \\ a; & x=\dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2}; & x>\dfrac{\pi}{2} \end{cases}$$

continuous at $x=\dfrac{\pi}{2}$.

13. If $f(x)=\begin{cases} \dfrac{1-\cos 10x}{x^2} & ; \quad x<0 \\ a & ; \quad x=0 \\ \dfrac{\sqrt{x}}{\sqrt{625+\sqrt{x}}-25} & \quad x>0 \end{cases}$

find the value of a which makes $f(x)$ continuous at $x=0$.

14. If $y=\sqrt{x}-\dfrac{1}{\sqrt{x}}$, show that $2x.\dfrac{dy}{dx}+y=2\sqrt{x}$.

15. If $y=\sqrt{\dfrac{x}{2}}+\sqrt{\dfrac{2}{x}}$, prove that $2xy\dfrac{dy}{dx}=\dfrac{x}{2}-\dfrac{2}{x}$.

16. If $y=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+.........+\dfrac{x^n}{n!}$, show that $\dfrac{dy}{dx}+\dfrac{x^n}{n!}=y$.

17. If $y=\sqrt{x}+\dfrac{1}{\sqrt{x}}$, show that $2x.\dfrac{dy}{dx}+y=2\sqrt{x}$.

18. Differentiate the following functions with respect to x.

i. $x^4.\sin x$ ii. $2^x \ln x$ iii. $x^3.\sin^{-1} x$ iv. $x^3.\cos x+e^x.\cot x$

v. $3^x \ln x+(1+x^2).\tan^{-1} x$ vi. $x^3 \sin x+2x\cos x-\cos ecx.\ln x$ vii. $x\sin x.\ e^x$

viii. $(x\sin x+\cos x)(x\cos x-\sin x)$ xi. $x\sin x\ \ \ln x$ x. $a^x \log_a^x+e^x.\ln x$

19. i. If $y=x\tan x$, prove that $x.\sin^2 x\dfrac{dy}{dx}=y^2+y\sin^2 x$.

ii. If $y=x\sin x$, prove that $\dfrac{1}{y}.\dfrac{dy}{dx}=\dfrac{1}{x}+\cot x$.

20. Differentiate the following functions with respect to x.

i. $y=\dfrac{x\sin x-\cos x}{x\cos x-\sin x}$ ii. $y=\dfrac{\sin x+x\cos x}{x\sin x-\cos x}$

iii. $y=\dfrac{e^x(x-1)}{x^2+1}$ iv. $y=\dfrac{\sin x+\ln x}{a^x+e^x}$

21. If $y=\dfrac{x}{x+5}$, prove that $x.\dfrac{dy}{dx}=y(1-y)$.

22. If $f(x)=\dfrac{1-\cos x}{1+\cos x}$, find $f'\left(\dfrac{\pi}{2}\right)$.

23. If $f(x)=\dfrac{x\sin x}{\cos x-\sin x}$, find $f'(0)$

24. If $f(t)=\dfrac{e^t.\sec t-\tan t}{1+\tan t}$; find $f'\left(\dfrac{\pi}{4}\right)$.

25.Differentiate the following functions with respect to x.

i. $y=\left[\dfrac{3+4x}{2-x}\right]^2$ ii. $y=\sqrt{\sin x^3}$ iii. $y=\tan(\ln x)$ iv. $y=\ln(\ln x)$

v. $y=\ln(\cos\sqrt{x})$ vi. $y=e^{\sin\sqrt{x}}$ vii. $y=\cos^2 x.\, e^{\tan x}$ viii. $y=\ln\left[x+\sqrt{1+x^2}\right]$

xi. $y=\sqrt{a^2+\sqrt{a^2+x^2}}$

26.Differentiate the following functions.

i. $y=\ln\sqrt{\dfrac{1-\cos x}{1+\cos x}}$ ii. $y=\left[x+\sqrt{1+x^2}\right]^n$ iii. $y=\sin(\sin(\ln 3x))$

iv. $y=\sin(e^x.\ln x)$ v. $y=\cos\sqrt{\sin\sqrt{x}}$ vi. $y=\sin\left[\cos(\tan\sqrt{ax})\right]$

27. i.If $y=\ln\sqrt{\dfrac{1+\tan x}{1-\tan x}}$, prove that $\dfrac{dy}{dx}=\sec 2x$.

ii.If $y=\left(x+\sqrt{x^2+4}\right)^3$, prove that $\dfrac{dy}{dx}=\dfrac{3y}{\sqrt{x^2+4}}$ iii.If $y=\sqrt{\dfrac{1-\sin 2x}{1+\sin 2x}}$, prove that $\dfrac{dy}{dx}+\sec^2\left(\dfrac{\pi}{4}-x\right)=0$

iv.If $y=\sqrt{\dfrac{1-x}{1+x}}$, prove that $(1-x^2)\dfrac{dy}{dx}+y=0$

v. If $x\sqrt{1+y}+y\sqrt{1+x}=0$, for $x\neq y$.

prove that $\dfrac{dy}{dx}=-\dfrac{1}{(1+x)^2}$

28.i.prove that $\dfrac{d}{dx}\left[\dfrac{x}{2}\sqrt{a^2-x^2}+\dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}\right]=\sqrt{a^2-x^2}$

ii. prove that $\dfrac{d}{dx}\left[2x.\tan^{-1}x-\ln(1+x^2)\right]=2\tan^{-1}x$

iii.If $y=\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}+\ln\sqrt{1-x^2}$, prove that $\dfrac{dy}{dx}=\dfrac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$

29. Differentiate the following functions.

i. $y=\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$ ii. $y=\tan^{-1}\left(\dfrac{1-\cos x}{\sin x}\right)$ iii. $y=\sin^{-1}\left[\dfrac{\sin x+\cos x}{\sqrt{2}}\right]$

iv. $y=\cos^{-1}\left[\dfrac{\sqrt{3}\cos x+\sin x}{2}\right]$ v. $y=\sin^{-1}\left[\dfrac{2x}{1+x^2}\right]$

30.i. $y=\sin^{-1}(\cos x)$ ii. $y=(\tan^{-1}x)^2$ iii. $y=\tan^{-1}(\ln x)$

iv. $y=(\sin^{-1}x^3)^3$ v. $y=\tan^{-1}(\sin x^2)$ vi. $y=\ln(\cos^{-1}x^3)$

31.i. $y=\cos^{-1}\left[\dfrac{x+\sqrt{1-x^2}}{\sqrt{2}}\right]$ ii. $y=\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-1}{x}\right]$ iii. $y=\sin\left[2\tan^{-1}\dfrac{\sqrt{1-x}}{\sqrt{1+x}}\right]$

32. If $y=\sin^{-1}\left[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right]$ prove that $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{2\sqrt{x-x^2}}$.

33. Differentiate the following functions.

i. $e^y=\tan(x+y)$ ii. $e^{x+y}=\sin(x^2+y^2)$

iii. $x+y=\tan^{-1}(xy)$ iv. $x^3y^3=\ln(x+y)+\sin(e^x)$

34. i.If $y=x\sin(a+y)$; prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin(a+y)-y\cos(a+y)}$

ii. If $\ln(xy)=x^2+y^2$, prove that $\dfrac{dy}{dx}=\dfrac{y(2x^2-1)}{x(1-2y^2)}$

iii. If $xy=1$, prove that *a.* $\dfrac{dy}{dx}+y^2=0$ *b.* $\dfrac{dx}{dy}+x^2=0$.

iv. If $y\ln x=x-y$, prove that $\dfrac{dy}{dx}=\dfrac{\ln x}{(1+\ln x)^2}$.

35. Differentiate the following functions.

i. $y=x^{\sin x-\cos x}$ ii. $y=(\sin x)^{\tan x}+(\cos x)^{\sec x}$ iii. $y=x^{\cos^{-1}x}$ iv. $y=(\ln x)^{\cos x}$ v. $y=(\sin x)^x+\sin^{-1}\sqrt{x}$.

36.Find $\dfrac{dy}{dx}$ of the following parametric functions.

i. $x=\dfrac{3at}{1+t^2}$, $y=\dfrac{3at^2}{1+t^2}$

ii. $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ iii. $x=\ln t$, $y=\sin t$ iv. $x=a\cos^2\theta$, $y=b\sin^2\theta$ v. $x=\sin t$; $y=\cos 2t$

37. i.If $x=\sin^{-1}\left(\dfrac{2t}{1+t^2}\right)$; $y=\tan^{-1}\left(\dfrac{2t}{1-t^2}\right)$, prove that $\dfrac{dy}{dx}=1$

ii.If $x=a\sin 2\theta(1+\cos 2\theta)$ and $y=a\cos 2\theta(1-\cos 2\theta)$, show that $\dfrac{dy}{dx}=\tan\theta$.

iii.If $x^3+y^3=t-\dfrac{1}{t}$ and $x^6+y^6=t^2+\dfrac{1}{t^2}$ show that $x^4y^2\dfrac{dy}{dx}=1$.

38. i..Differentiate $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-1}{x}\right]$ $w.r.t$ $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

ii.. Differentiate $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ $w.r.t$ $\tan^{-1}x$.

39.i.If $x=\ln t$, and $y=\dfrac{1}{t}$, show that $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0$.

ii.If $x=a\sin t-b\cos t,\ y=a\cos t+b\sin t$ show that $\dfrac{d^2y}{dx^2}=-\dfrac{x^2+y^2}{y^3}$

20.*i*.Show that $\lim_{x\to\frac{\pi}{6}}\left[\dfrac{\sqrt{\dfrac{\pi^2}{16}\left[x^2-\dfrac{\pi^2}{36}\right]}}{\tan\sqrt{\dfrac{\pi}{4}\left(x-\dfrac{\pi}{6}\right)}}-\dfrac{\sin\left(x-\dfrac{\pi}{6}\right)}{\sqrt{x}-\sqrt{\dfrac{\pi}{6}}}\right]=\dfrac{\pi\sqrt{\pi}}{3\sqrt{2}}$

ii.Find $\lim_{x\to 0}\dfrac{1-\cos^2(3\sin x)}{1-\cos 2x}$

iii.The tangents drawn to the curve $y^2=x(2-x)^2$ at the point (1, 1) intersect again the curve at the point *P*, find the coordinates of *P*.

21.*a*.Using the first principles, prove that $\dfrac{d}{dx}(\tan x)=\sec^2 x$. Deduce that $\dfrac{d}{dx}(\tan^{-1}x)=\dfrac{1}{1+x^2}$. Show also that when y is a function of x, $(1+x^2)\dfrac{dy}{dx}=\dfrac{dy}{d(\tan^{-1}x)}$.

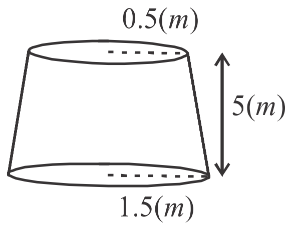
Let $y=\dfrac{x}{\sqrt{1+x^2}}+\cos\left[2\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}\right]$. Using the substitution $\tan^{-1}x=\theta$, or otherwise show that

$$\frac{dy}{d(\tan^{-1}x)}=\sqrt{2}\cos\left[\frac{\pi}{4}+\tan^{-1}x\right].$$

b.Let $f(x)=\dfrac{1+2x}{x(x+1)}$. Sketch the graph of $y=f(x)$. Hence by considering the line $y=mx$. show that the eqation $mx^3+mx^2-2x-1=0$ has *i*. 3 real roots when $m > 0$

ii.Only one real root when $m \leq 0$.

c.A straight tree trunk in the shape of a frustum is shown in the adjoining diagram.
Show that the height of the cylinder with maximum volume that can be cut from this tree trunk is 2.5(*m*).



22.*a*. Show that $\lim_{x \to \frac{\pi}{4}} \frac{256x^4 - \pi^4}{\tan x - 1} = 8\pi^3$.

b.The tangents drawn to the curve $x^2 = 4y$ at the points $P(2t, t^2)$ and $Q(4t, 4t^2)$ meet at *R*. Show that the locus of the point *R* is given by $2x^2 = 9y$ Here $t\,(\neq 0)$ is a parameter.

23.*a*.Show that $\lim_{x \to 0} \frac{x\sin(x^2)}{\sin x(1 - \sqrt{\cos x})} = 4$

b.Let *C* be the curve with the equation $x^3 - y^2 = 0$. Show that the equation of the tangent drawn to *C* at $P(4t^2, 8t^3)$ is $3tx - y - 4t^3 = 0$. where $t\ (\neq 0)$ is a real parameter. If this tangent is perpendicular to the curve *C* at $Q\ (4T^2, 8T^3)$ Show that $T = -\frac{1}{9t}$.

24.*i*.Find $\lim_{x \to \frac{\pi}{3}} \frac{\tan^3 x - 3\tan x}{\cos\left(x + \frac{\pi}{6}\right)}$.

ii.A curve *C* is given by the equation $y = \frac{\sin x}{\cos^2 x} + \ln\left[\frac{1 + \sin x}{\cos x}\right]$. The gradient of the tangent drawn to *C* at a point *P* is 16. Show that $P = \left[\frac{\pi}{3}, 2\sqrt{3} + \ln(2 + \sqrt{3})\right]$ Here $0 < x < \frac{x}{2}$.

iii.Find $\lim_{x \to 2} \frac{\sqrt{2x + \sqrt{x - 1}} - \sqrt{5}}{x - 2}$

iv.If $\lim_{n \to \infty} \frac{n}{a}\sqrt{a - a\cos\left(\frac{2\pi}{n}\right)} = \pi$. Find *a*. where $a > 0$

25.*i*.Show that $\lim_{x \to 0} \left[\frac{(27 + x)^{\frac{1}{3}} - 3}{x^2}\right] . \sin 9x = \frac{1}{3}$

ii.*a*.Let $f(x) = 1 + \frac{1}{x^2 - 2x}$ show that $f'(x) = \frac{-2(x - 1)}{x^2(x - 2)^2}$.

Sketch the graph of $y = f(x)$, indicating turning points and asymptotes. Hence draw the graph of $y = \frac{1}{f(x)}$

b.A wire of length *l* is bent into the shape of an isosceles triangle. Show that the area of this triangle will be maximum when it is an equilateral triangle and find its maximum area.

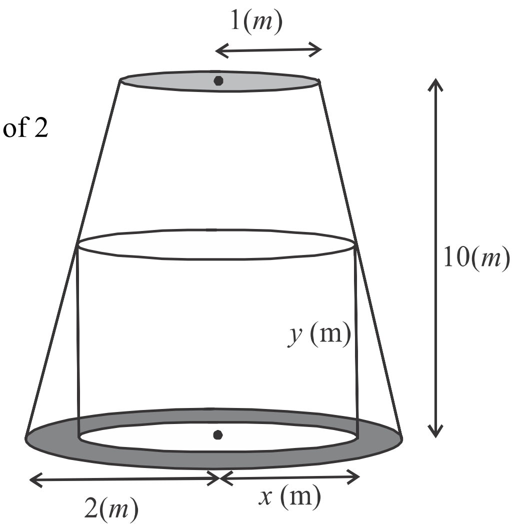
26.*a*.Let $f(x)=\dfrac{(2x-1)^2}{(x-1)^2}$ for $x\neq 1$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=-\dfrac{2(2x-1)}{(x-1)^3}$ for $x\neq 1$.

Sketch the graph of $y=f(x)$ indicating the asymptotes, y -intercept and the turning points.

It is given that $f''(x)=\dfrac{8\left(x-\frac{1}{4}\right)}{(x-1)^4}$ for $x\neq 1$. Find the x coordinate of the point of inflection of the graph of $y=f(x)$.

b.A base of a wooden cone is given with the height of 10*m* and circular radii of 2 (*m*), 1 (*m*). If a cylindrical pole with radius *x m* and height *y* m has been cut from the wooden block, then show that $y=10(2-x)$. If the volume of the cylinder is Vm^3, show that $V=10\pi x^2(2-x)$. Hence show that the radius of the cylinder which can be cut from the wooden block with maximum volume is $\dfrac{4}{3}m$.



27.*a*.Let $f(x)=\dfrac{(x+1)(2x-1)}{(x-2)^2}$ for $x\neq 2$.

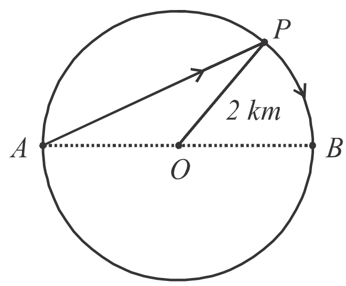
Show that the $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{-9x}{(x-2)^3}$ for $x\neq 2$.

Hence, find the coordinates of turning point of the curve $y=f(x)$ and distinguish whether it is maximum or minimum.

It is given that $f''(x)=\dfrac{18(x+1)}{(x-2)^4}$ for $x\neq 2$. Find the coordinates of inflection point of $y=f(x)$.

Draw the rough sketch of $y=f(x)$ indicating the asymptotes, turning point and inflection point.

b.Figure shows a circular lake with center *O* and radius 2*km.* An animal which is capable of swimming at a uniform speed of $3kmh^{-1}$ and running at a uniform speed of $4kmh^{-1}$ needs to travel from *A* to *B*, decides to swim from *A* to *P* and then runs from *P* to *B* along the lake side. The total time taken by the animal to reach *B* from *A* is T hours. Let $\measuredangle PAB=\theta$ radians; Here $0<\theta<\dfrac{\pi}{2}$.



i. Show that $T=\dfrac{1}{3}(4\cos\theta+3\theta)$

ii. Show that $\dfrac{dT}{d\theta}=\dfrac{1}{3}(3-4\sin\theta)$ and find the value of θ for which *T* attains its maximum.

28.*i.* Show that $\lim_{x\to\frac{\pi}{3}} \dfrac{\sqrt{3}\cos x - \sin x}{\left(\sqrt{\frac{\pi}{3}} - \sqrt{x}\right)(\cos x + \sqrt{3}\sin x)} = 2\sqrt{\frac{\pi}{3}}$

ii.a. Let $f(x) = \dfrac{x^2 - 6x + 4}{x^2 + 2x + 4}$ show that $f'(x) = \dfrac{8(x-2)(x+2)}{(x^2 + 2x + 4)^2}$ and deduce that the curve has turning points at (-2,5) and $\left(2, -\frac{1}{3}\right)$. Draw a rough sketch of the curve $y = f(x)$ showing the turning points and asymptotes.

Hence, find the number of solutions of the equation $(x^2 - 6x + 4) = (x^2 + 2x + 4).(e^x - e^{-x})$

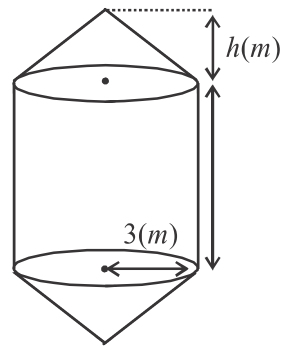
b. The total surface area of a closed right circular cylinder is $2\pi(m^2)$. Show that its volume is $V = \pi(r - r^3)(m^3)$ Here r is the radius of the cylinder. When r varies, show that the maximum volume of the cylinder is $\dfrac{2\pi}{3\sqrt{3}} m^3$.

29.*a.* Let $y = \dfrac{(x-2)^2}{x^2 + 4}$ for $x \in \mathbb{R}$. Show that $0 \le y \le 2$.

Draw the curve $y = \dfrac{(x-2)^2}{x^2 + 4}$, showing the turning points and asymptotes. The equation $x(x^2 + 4) = (x-2)^2$ has only one real root. Explain why?

b. A container is made by rigidly joining two hollow cones of radius 3 meters and height h meters with a hollow right circular cylinder of same radius and height H meters as shown in the diagram. The total volume of the container is $900m^3$. Show that $H = \dfrac{100}{\pi} - \dfrac{2}{3}h$.

If the total surface area of the container is Sm^2, show that $S = 600 - 4\pi h + 6\pi\sqrt{9 + h^2}$.
Find the value of h for which S is minimum.



30.*i.* Show that $\lim_{x\to 0} \dfrac{1 - \cos x \cos 2x \cos 3x}{\sin x \sin 2x} = \dfrac{7}{2}$.

ii. A curve C in the *xv* plane is given in terms of a real parameter θ, by the equations $x = 1 + \sin\theta, y = 3 + \cos 2\theta$.
Find the derivative $\dfrac{dy}{dx}$ in terms of θ and find the coordinates of the point where the normal drawn at the point A intersect the curve again when $\theta = \dfrac{\pi}{6}$.

31. *i.* If $\lim_{x\to 1} \dfrac{1 - \sqrt{x}}{\left[\frac{1}{a}\cos^{-1} x\right]^2} = \dfrac{9}{4}$; Here a is positive constant . Show that $a = 3$.

ii.. A curve C in the *xy* plane is given in terms of a real parameter θ, by the equation $x = 1 + \tan\theta,\ y = 2 + \cos 2\theta$.

Find the derivative $\frac{dy}{dx}$ in terms of θ , and show that the normal drawn at the point *A* do not intersect the curve again when $\theta = \frac{\pi}{4}$.

32.*a*.Find the constants *a* and *b* such that $\lim_{x \to \infty} \left\{ \frac{x^2+1}{x+1} - ax - b \right\} = 0$.

b.A function *f*(*x*) is defined by $f(x) = \begin{cases} \frac{x^2}{\alpha} & ; 0 \le x < 1 \\ \alpha & ; 1 \le x < \sqrt{2} \\ \frac{2\beta^2 - 4\beta}{x^2} & ; \sqrt{2} \le x < \infty \end{cases}$

If the function *f* is continuous for every $x \in [0, \infty)$, then find the constants of α and β. Where α, $\beta \in Z$ and $\alpha \ne 0$

c. Let $y = (1 + \sin^{-1} x).\sin^{-1} x$

Show that $(1 - x^2)\frac{d^2y}{dx^2} - x.\frac{dy}{dx}$ is independent of *x*.Hence , for $n = 1,2,3,4$ find $\frac{d^n y}{dx^n}$ $at\ x = 0$.

33.*a*.Let $f(x) = \frac{p - x}{(x+q)^2}$; where $x \in \Re$ with $x \ne -q$ and *p* and *q* are constants. The curve *C* with the equation $y = f(x)$ has an asymptote with the equation $x = 2$ and has a turning point at $x = 6$.

Determine the values of *p* and *q*. Show that $x = 6$ is the only turning point for the curve *C*.

Write down equations of other asymptotes if any for *C*.

If is given that $f''(x) = \frac{2(3p + 2q - x)}{(x+q)^4}$ for $x \ne q$; where $f''(x)$ denotes the second derivative of $f(x)$. Sketch the graph of $y = f(x)$, clearly indicating the turning point, asymptotes and points of inflection.

find the number of solutions of the equation $(p - x)e^x - (x+q)^2 = 0$.

b.A trapezium *ABCD* is such that *AD* and *BC* are parallel and $AB = BC = CD = a$.

Let *s* be the area of the trapezium *ABCD* and $D\hat{A}B = C\hat{D}A = \theta$.

Express *s* as a function of θ.

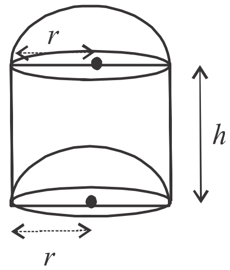
Show that $\frac{ds}{d\theta} = 0$ if and only if $\theta = \frac{\pi}{3}$.

By considering the sign of $\frac{ds}{d\theta}$, in appropriate intervals of θ find the maximum possible area of the trapezium.

34.a.Let $f(x)=\dfrac{2x^2}{(x+2)(x-4)}$; where $x\neq -2,\,4$ show that $f'(x)=\dfrac{-4x(8+x)}{(x+2)^2(x-4)^2}$.

Hence find the stationary points of $y=f(x)$ and sketch the graph of $f(x)$ indicating the asymptotes. Using the graph, find the range of values of k , of the equation $2x^2-k(x+2)(x-4)=0$ so that the equation has no real roots for x.

b.A hollow object is to be made by fixing two hollow hemispheres to two opened ends of a right circular hollow cylinder as shown in the figure. The radius of hemisphere is r and that of the cylinder is also r.



If the compositue body is to be made using minimum amount of metal and having volume $4000\pi(cm^3)$, find the height and radius of the cylinder. Also find the area of the metal sheet required.

35.a. Show that $\lim\limits_{x\to\frac{\pi}{3}} \dfrac{\sin\left(\frac{\pi}{3}-x\right)}{2\cos x-1}=\dfrac{1}{\sqrt{3}}$

b. Using the first principles , find the derivative of $\sqrt{1-2x^2}$ with respect to x.

c. Differentiate the following functions with respect to x. (simphfication is not necessary)

i. $y=\tan^{-1}\left(\dfrac{1-\sin x}{1+2\sin x}\right)$ ii. $y=e^{ax}\cos bx$ iii. $y=\ln\left|\dfrac{a+b\cos x}{b+a\cos x}\right|$

d. If $y=\sqrt{4+\sin^2 x}$ Show that

i. $y.\dfrac{dy}{dx}=\dfrac{1}{2}\sin 2x$ ii. $y.\dfrac{d^3y}{dx^3}+3\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)+4y\dfrac{dy}{dx}=0$

Evaluate $\left[\dfrac{d^3y}{dx^3}\right]_{x=0}$

36.a. Let $f(x)=\dfrac{(x+2)}{(2x+1)^2}$; for $x\neq -\dfrac{1}{2}$

Show that $f'(x)=-\dfrac{(2x+7)}{(2x+1)^3}$,for $x\neq -\dfrac{1}{2}$

where $f'(x)$ denotes the first derivative of $f(x)$. It is given that

$f''(x)=\dfrac{8(5+x)}{(2x+1)^4}$,for $x\neq -\dfrac{1}{2}$.

where $f''(x)$ denotes the second derivative of $f(x)$.Sketch the graph of $y=f(x)$ indicating clearly stationary points,asymptotes, and points of inflections.

b.A closed container in the shape of a right circular cylinder of volume $256\pi(cm^3)$ is to be formed. The material cost per unit area of curved surface of the container is twice that for the flat surfaces. The cost of material per unit area of flat surfaces is $100(Rs)$. If the height of the container is $h\ (cm)$ and radius of flat surfaces is $r(cm)$.

Show that the total cost C making this cylindrical container is given by $C=\frac{102400}{r}.\pi+200\pi r^2$.

Hence find the value of r which minimises C and minimum C.

37.*a*.Let $x=\cot(kt)$ and $y=e^{\operatorname{cosec}(kt)}$, where $k\in\Re$.

Show that $(1+x^2)\frac{d^2y}{dx^2}+x.\frac{dy}{dx}-\sqrt{1+x^2}.\left(x\frac{dy}{dx}+y\right)=0$

b.Let $y=\frac{x-2}{(x-1)^2}$, for $x\neq 1$

Find

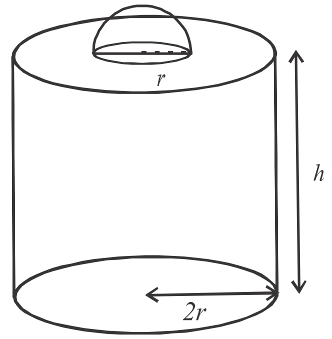
i. asymptotes

ii. stationary poins

iii.points of inflection

Showing clearly the above properties sketch a rough graph of $y=f(x)$. Also , deduce the number of solutions of the equation $(x-4)(x-1)^2-2(x-2)=0$.

c.A casket shown in the figure is to be formed with thin tin. The casket consists of a hollow hemisphere of radius r and hollow cylinder of radius $2r$. The hemispherical part is joined to the upper flat surface of the cylinder by removing common area for both from the upper flat surface of the cylinder.



If the area of casket is A, show that $3r<\sqrt{\frac{A}{\pi}}$. Also find the values of r and h which maximise the volume of the casket.

d.A function $f(x)$ is defined as
$$f(x)=\begin{cases}\frac{\sin(a+1)x+\sin x}{x} & ;\ \text{for } x<0\\ c & ;\ \text{for } x=0\\ \frac{(x+bx^2)^{\frac{1}{2}}-x^{\frac{1}{2}}}{bx^{\frac{3}{2}}} & ;\ \text{for } x>0\end{cases}$$

where $a,b,c,\in\Re$. Find the values of a,b, and c such that the function $f(x)$ is continuous at $x=0$.

38.*a*.Let $f(x)=\dfrac{x^2+x+1}{x^2-x+1}$.

Show that $f'(x)=\dfrac{2(x^2-1)}{(x^2-x+1)^2}$

By considering the sign of $f'(x)$, sketch the graph of $y=f(x)$. Deduce the graph of $y=\dfrac{1}{f(x)}$.

Hence find the real root of the equation $\left[f(x)\right]^2=1$ Also deduce the graph of $y=f(-x)$.

b.Let $Q=\left[a\cos\theta,\ a\sin\theta\right]$ be a point. Show that *Q* lies on the circle with the equation $x^2+y^2=a^2$ .Show that the equation of the tangent drawn to the above circle at *Q* is $x\cos\theta+y\sin\theta=a$. Let *R* be the base of the perpendicular drawn from the point *P*(*a*,0) to the above tangent drawn at *Q*. If the area of the triangle *PQR* is *S*,

Show that $S=\frac{1}{2}a^2(1-\cos\theta).\sin\theta$

Also show that $\dfrac{dS}{d\theta}=a^2\sin\dfrac{\theta}{2}.\sin\dfrac{3\theta}{2}$ and the maximum value of S is $\dfrac{3\sqrt{3}}{8}a^2$ in the range $\dfrac{\pi}{2}<\theta<\pi$.

39.Let $f(x)=\dfrac{x^2-1}{x^2-4}$; for $x\neq\pm 2$.

Show that the derivative $f'(x)$ of *f*(*x*) is given by $f'(x)=-\dfrac{6x}{(x^2-4)^2}$ for $x\neq\pm 2$. By indicating asymptotes, turning points, and y- intercepts , sketch the graph of $y=f(x)$.

Hence find the values of *k* , such that the equation $(k-1)x^2-4k+1=0$.

a. has no at least a real solution

b. has real coincident roots

c. has distinct real roots.

40.*a*.Let $f(x)=\dfrac{3x^2-8x-12}{3(x-2)^3}$. for $x\neq 2$.

Show that the derivative $f'(x)$ of *f*(*x*) is given by $f'(x)=\dfrac{3x^2+4x+52}{3(x-2)^4}$.

Sketch the graph of $y=f(x)$ indicating asymptotes, turning points and points of inflection. It is given that the second derivative $f''(x)$ of *f*(*x*) is

$$f''(x)=\frac{2(x^2-36)}{(x-2)^5}.$$

b.In making a cylindrical water tank the cost per unit area of the base is *a* (*Rs*) and that for circular surface is *b* (*Rs*) . The total amount can be spent in making the tank is *c* (*Rs*) . where *a, b* and c are constants. If the internal radius of the tank is x , show that the volume *v* of the tank is given by $v=\dfrac{cx-\pi a x^3}{2b}$

Show that when v is maximum , the expense for the base is $\frac{c}{3}(Rs)$.

41.a.Let $f(x)=\frac{(x+1)}{(x-3)^2}$, for $x \neq 3$,

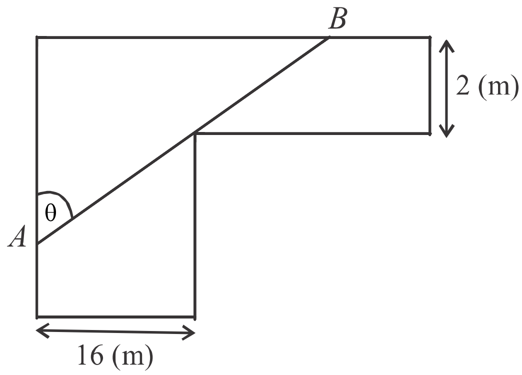
Find the first derivative of $f(x)$. It is given that the second derivative of $f(x)$, as $f''(x)=\frac{2(x+9)}{(x-3)^4}$.

Sketch the graph of $y=f(x)$ indicating clearly , turning points, asymptotes, and points of inflection. Hence determine the number of solutions for the equation $f(x)-1=0$.

b.A rod AB is taken from a hall of width 16(m) to a passage way of width 2 (m) as shown in the figure.

If the rod is always kept horizontal , obtain an expression for the length of the rod AB in terms of θ, so that it can take through the turning of the passage way.

Hence show that the maximum length of the rod which can take is $10\sqrt{5}(m)$.



42.a.If $y=(1+\sin^{-1}x)\sin^{-1}x$, show that $(1-x^2).\frac{d^2y}{dx^2}-x.\frac{dy}{dx}$ is indepedent of x.

b.Let $f(x)=\frac{2x+3}{(x-1)^3(x+1)}$; for $x \in \Re \{-1,1\}$.

Show that $f'(x)=\frac{-2(3x+2)(x+2)}{(x-1)^4(x+1)^2}$.

Sketch the graph of $y=f(x)$ indicating clearly turning points, asymptotes and points of intersection of the axes.

c. A tent is to be formed to have a given volume and given base radius so that its upper part is a cone and lower parts is a cylinder. Show that in forming the tent the amount of canvas required is minimum when the semi- vertical angle of the cone is $\cos^{-1}\left(\frac{2}{3}\right)$.

43.a.If $\sin y = x\sin(p+y)$, show that $\sin p\frac{dy}{dx}-\sin^2(p+y)=0$, where p is a constant.

Show also that $x^2.\sin p.\frac{d^2y}{dx^2}+(2x\sin p-\sin 2y)\frac{dy}{dx}=0$

b.Let $f(x)=\frac{5x}{(x-2)^2}$, for $x \neq 2$.

Find the first derivative of $f(x)$

It is given that $f''(x)=\frac{10(x+4)}{(x-2)^4}$.

Sketch the graph of $y = f(x)$ indicating turning points , asymptotes and points of inflection. Obtain the equation of the tangent drawn to the curve at the point $\left(-4, \frac{-5}{9}\right)$. Also sketch this tangent on the graph.

c.A trapizium has three sides each of length 10(*cm*) except for the base. Find the length of the base when the area of the trapizium is maximum. Also find the maxium area.

44.*a*.Let $f(x) = \dfrac{2x^2}{(x-2)(x-6)}$ for $x \in \Re - \{2,6\}$.

Show that $f'(x) = \dfrac{4(3-x)}{x^3}\left[f(x)\right]^2$.

Sketch , the graph of $y = f(x)$ indicating clearly , turning points, asymptotes.

Hence show that the equation $x^2 - 9x + 4 = -12x^{-1}$ has three roots.

b.A thin rectangular sheet *ABCD* with $AB = 1(m)$ is folded such that the point *D* falls on the side *BC* folding along the line *PQ*, where *P* and *Q* lie on the sides *AD* and *DC* respectively.

If $PQ = x(m)$ and $P\hat{Q}D = \theta$, obtain an expression for *x* in terms of θ.

If $t = \dfrac{1}{x}$, show that $t = 2\sin^2\theta.\cos\theta$

Hence find the value of θ, which maximise *t*. show that when $\theta = \sin^{-1}\sqrt{\dfrac{2}{3}}$ the length of *PQ* becomes minimum.

Also , show that the minimum length of *PQ* is $\dfrac{3\sqrt{3}}{4}\ (m)$.

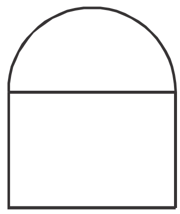
45.*a*.Let $f(x) = \dfrac{9(x^2+2x-4)}{(x+2)^3}$; for $x \neq -2$.

Show that the derivative $f'(x)$ of $f(x)$ is given by $f'(x) = \dfrac{-9(x-4)(x+4)}{(x+2)^4}$, for $x \neq -2$.

Given that $f''(x) = \dfrac{18(x^2-2x-32)}{(x+2)^5}$, for $x \neq -2$.

Find the *x* - coordinates of the points of inflection of the graph $y = f(x)$. Sketch the graph of $y = f(x)$; indicating clearly asymptotes, turning points and y intercepts.

b.A window shown in the figure consisting a rectangle and semi- circular arch is to be formed using given length of steel rail. The upper part of the window is covered with colour glass and rectangular part is covered by a clear glass. The amount of light passes through clear glass is 3 times as that of colour glass. show that in order to pass maximum light through the window the ratio of the length and breath of the rectangle is $\dfrac{6}{\pi+6}$.



46.a.Let $f(x)=\dfrac{(x-1)(2x+1)}{(x+1)^2}$, for $x\neq 1$. Show that the derivative $f'(x)$ of $f(x)$ is given by $f'(x)=\dfrac{5x+1}{(x+1)^3}$, for $x\neq -1$.

Given that the second derivative $f''(x)$ of $f(x)$ is $f''(x)=\dfrac{2(1-5x)}{(x+1)^4}$.

Sketch the graph of $y=f(x)$ indicating clearly asymptotes , turning points and points of inflection.

b.A petagon is formed using a fine wire of length a, such that the part $BCDE$ is a rectangle and ABE part is an equilat eral triangle. If $CD=x$, show that the area of the pentagon is $\dfrac{x}{4}\left[2a-(6-\sqrt{3})x\right]$. Also show that the maximum area of the pentagon is $\dfrac{a^2}{4(6-\sqrt{3})}$.

47.a.Let $f(x)=\dfrac{(x+2)}{(x+1)^2}$, for $x\neq -1$.

Show for $x\neq -1$ that $f'(x)=\dfrac{-(x+3)}{(x+1)^3}$

Given that $f''(x)=\dfrac{2(x+4)}{(x+1)^4}$, for $x\neq -1$.

Sketch the graph of $y=f(x)$ indicating clearly y- intercepts , turning points, asymptotes and points of inflection.

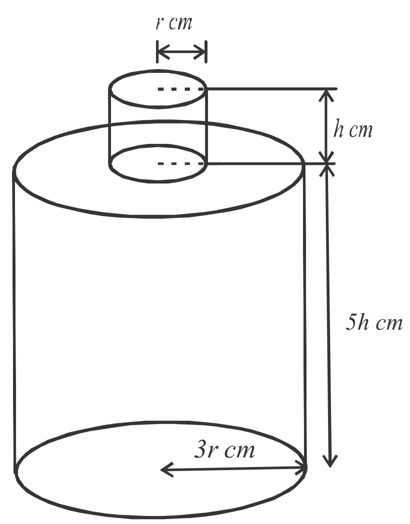
b. A steel wire of length 50 (m) is cut in to two parts and by using them a rectangle and a circle are formed. Find the length of the parts of the wire so as to minimise the whole area.

48.a.Let $f(x)=\dfrac{16(x-1)}{(x+1)^2(3x-1)}$ for $x\neq -1, \dfrac{1}{3}$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{-32x(3x-5)}{(x+1)^3(3x-1)^2}$ for $x\neq -1, \dfrac{1}{3}$.

Sketch the graph of $y=f(x)$ indicating the asymptotes and the turning points.Using the graph, find the values of $k\in\Re$ such that the equation $k(x+1)^2(3x-1)=16(x-1)$ has exactly one root.

b. A bottle with a volume of $391\pi cm^3$ is to be made by removing a disc of radius r cm from the top face of a closed hollow right circular cylinder of radiues $3r$ cm and height $5h$ cm, and fixing an open hollow right circular cylinder of radiues r cm and height h cm, as shown in the figure. It is given that the total surface area $S\,cm^2$ of the bottle is $S=\pi r(32h+17r)$. Find the value of r such that S is minimum.



49.a.Let $f(x)=\dfrac{9(x^2-4x-1)}{(x-3)^3}$ for $x\neq 3$.

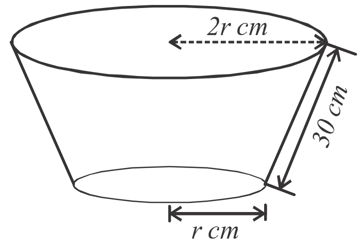
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=-\dfrac{9(x+3)(x-5)}{(x-3)^4}$ for $x\neq 3$.

Sketch the graph of $y=f(x)$ indicating the asymptotes, y-intercept and the turning points.

It is given that $f''(x)=\dfrac{18(x^2-33)}{(x-3)^5}$ for $x\neq 3$.

Find the x-coordinates of the points of inflection of the graph of $y=f(x)$.

b.The adjoining figure shows a basin in the form of a frustum of a right circular cone with a bottom. The slant length of the basin is 30 cm and the radius of the upper circular edge is twice the radius of the bottom. Let the radius of the bottom be r cm.



Show that the volume $V cm^3$ of the basin is given by

$V=\dfrac{7}{3}\pi r^2\sqrt{900-r^2}$ for $0<r<30$.

Find the value of r such that volume of the basin is maximum.

50..a.Let $f(x)=\dfrac{x(2x-3)}{(x-3)^2}$ for $x\neq 3$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{9(1-x)}{(x-3)^3}$ for $x\neq 3$.

Hence, find the interval on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing. Also, find the coordinates of the turning point of $f(x)$.

It is given that $f''(x)=\dfrac{18x}{(x-3)^4}$ for $x\neq 3$.

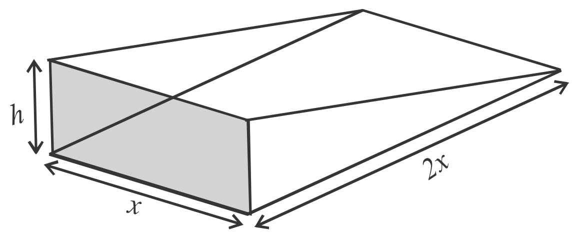
Find the coordinates of the point of inflection of the graph of $y=f(x)$.

Sketch the graph of $y=f(x)$ indicating the asymptotes, the turning point and the point of inflection.

b.The adjoining figure shows the portion of a dust pan without its handle. Its dimensions in centimetres, are shown in the figure.
It is given that its volume $x^2h\, cm^3$ is $4500cm^3$.

Its surface area S cm^2 is given by $S=2x^2+3xh$.

Show that S is minimum when $x=15$.



51.Let $f(x)=\dfrac{(2x-3)^2}{4(x^2-1)}$ for $x\neq 1$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{(2x-3)(3x-2)}{2(x^2-1)^2}$ for $x\neq \pm 1$.

Sketch the graph of $y=f(x)$ indicating the asymptotes, y-intercept and the turning points.

Using the graph, find all real values of x satisfying the inequality $\dfrac{1}{f(x)} \le 1$.

52.*a*.Let $f(x) = \dfrac{2x+3}{(x+2)^2}$ for $x \ne -2$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \dfrac{-2(x+1)}{(x+2)^3}$ for $x \ne -2$.

Hence, find the interval on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

Also, find the coordinates of the turning point of $f(x)$.

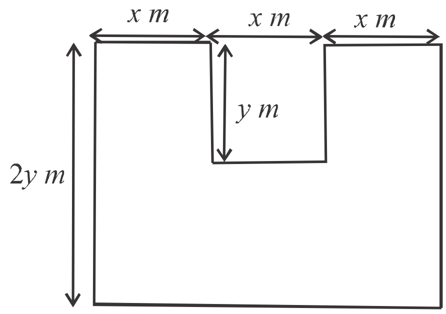
It is given that $f''(x) = \dfrac{2(2x+1)}{(x+2)^4}$ for $x \ne -2$. Find the coordinates of the point of inflection of the graph of $y = f(x)$.

Sketch the graph of $y = f(x)$ indicating the asymptotes, the turning point and the point of inflection.

State the smallest value of k for which $f(x)$ is one - one on $[k, \infty)$.

b.The shaded region shown in the figure is of area $45m^2$. It is obtained by removing a rectangle of length x *m* and width y *m* from a rectangle of length $3x$ *m* and width $2y$ *m*. Show that the perimeter L *m* of the shaded region is given by $L = 6x + \dfrac{54}{x}$ for $x > 0$.

Find the value of x such that L is minimum..



53.*a*. Let $f(x) = \dfrac{x^2}{(x-1)(x-2)}$ for $x \ne 1, 2$.

Show that $f'(x)$, the derivative of $f(x)$, is given by

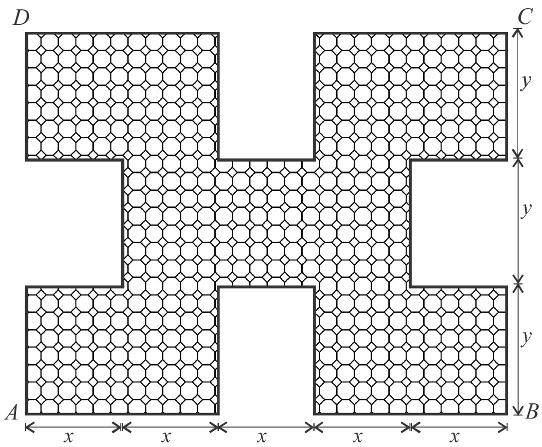
$$f'(x) = \frac{x(4-3x)}{(x-1)^2(x-2)^2} \quad \text{for } x \ne 1, 2.$$

Sketch the graph of $y = f(x)$ indicating the asymptotes and the turning points. Using the graph, solve the inequality $\dfrac{x^2}{(x-1)(x-2)} \le 0$.

b.The shaded region shown in the adjoining figure is of area $385\ m^2$. This region is obtained by removing four identical rectangles each of length y metres and width x metres from a rectangle *ABCD* of length $5x$ metres and width $3y$ metres. Show that $y = \dfrac{35}{x}$ and that the perimeter P of the shaded region, measured in metres, is given by

$$P = 14x + \frac{350}{x} \quad \text{for } x > 0.$$

Find the value of x such that P is minimum.



54.*a*.Let $f(x) = \dfrac{(x-3)^2}{x^2 - 1}$ *for* $x \neq \pm 1$.

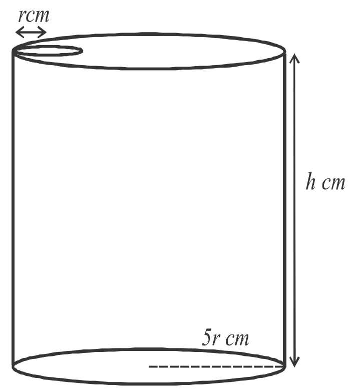
Show that $f'(x)$, the derivative of *f*(*x*), is given by $f'(x) = \dfrac{2(x-3)(3x-1)}{(x^2-1)^2}$

Write down the equations of the asymptotes of $y = f(x)$.

Find the coordinates of the point at which the horizontal asymptote intersects the curve $y = f(x)$. Sketch the graph of $y = f(x)$ indicating the

asymptotes and the turning points.

b.A thin metal container, in the shape of a right circular cylinder of radius 5*r cm* and height *h cm* has a circular lid of radius 5*r cm* with a circular hole of radius *r cm*. (See the figure .)

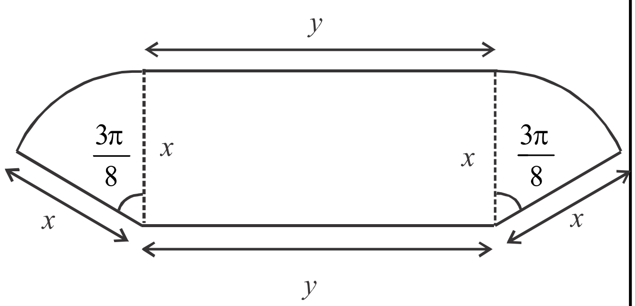


The volume of the container is given to be $245\pi\, cm^3$. Show that the surface area $S\, cm^2$ of the container with the lid containing the hole is given by $S = 49\pi\left(r^2 + \dfrac{2}{r}\right)$ for $r > 0$ Find the value of *r* such that *S* is minimum

55. *a*.Let $f(x) = \dfrac{4x+1}{x(x-2)}$ for $x \neq 0, 2$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = -\dfrac{2(2x-1)(x+1)}{x^2(x-2)^2}$ for $x \neq 0, 2$. Hence, find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$, is decreasing. Sketch the graph of $y = f(x)$ indicating the asymptotes, *x*-intercept and the turning points. Using this graph, find all real values of *x* satisfying the inequality $f(x) + |f(x)| > 0$.

b.The shaded region *S* of the adjoining figure shows a garden consisting of a rectangle and two sectors of a circle each subtending an angle $\dfrac{3\pi}{8}$ at the centre. Its dimensions, in metres, are shown in the figure. The area of *S* is given to be $36m^2$. Show that the perimeter *p m* of *S* is given by $p = 2x + \dfrac{72}{x}$ for $x > 0$ and that *p* is minimum `when $x = 6$.

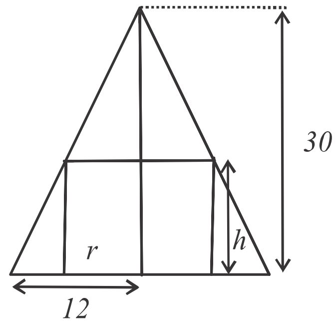


56.*a*.Let $f(x) = \dfrac{x}{(x-1)^2}$ for $x \neq 1$. Find the derivative of $f'(x)$ of $f(x)$. Hence find the increasing and decreasing interval of $f(x)$.

It is given that $f''(x) = \dfrac{2x+4}{(x-1)^4}$, for $x \neq 1$.

By indicating turning points, asymptotes and points of inflection sketch the graph of $y = f(x)$.

b.The diagram shows a solid cylinder of height h(*cm*) and radius *r* (*cm*) inscribed in a hollow cone of height 30(*cm*) and base radius 12(*cm*). The upper circular edge of the cylinder touches the cone.



i. Show that the volume of the cylinder is $V = \pi\left[30r^2 - \frac{5}{2}r^3\right]$.

ii.Find the volume of the biggest cylinder in the cone and show that its volume is $\frac{4}{9}$ of the volume of the cone.

57..*a*.For $x \neq 1$, let $f(x) = \frac{x(x+1)}{(x-1)^2}$.

Show that the derivative $f'(x)$ of $f(x)$ is given $f'(x) = \frac{-(3x+1)}{(x-1)^3}$ for $x \neq 1$.

Hence find the range of $f(x)$ increasing and decreasing. Also find the coordinates of the turning points. Its given that $f''(x) = \frac{6(x+1)}{(x-1)^4}$ for $x \neq 1$. Sketch the graph of $y = f(x)$ indicating turning points, asymptotes and points of inflection.

b.A wire of length *l*(*cm*) is cut in to two pieces and made a circle of circumference *x*(*cm*) and a square of a side $\frac{l-x}{4}$.

Show that total area *A*(*x*) is given by $A(x) = \frac{x^2}{4\pi} + \frac{(l-x)^2}{16}$. Show that the total area is minimum when the side of the square is $\frac{l}{\pi+4}$

58.a.For $x \neq -2$, let $f(x) = \frac{(x-1)(3x+1)}{(x+2)^2}$.

Show that the derivative $f'(x)$ of $f(x)$ is given by $f'(x) = \frac{2(7x-1)}{(x+2)^3}$ for $x \neq -2$.

Hence find the range of *x*, for $f(x)$ increasing and decreasing. It is given that , $f''(x) = \frac{2(17-13x)}{(x+2)^4}$ for $x \neq -2$.

By indicating asymptotes, turning points and points of inflection sketch the graph of $y = f(x)$.

b.A printer in printing a document keep a space of $\frac{1}{2}$ units from the top, bottom and right edge and 1(unit) from left edge. If the total area of the paper is 96 (square units), when the printed area is maximum find the length and the breadth of the page.

59. i. If $y = \tan x + \sec x$ prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$

ii. If $y = ae^{mx} + be^{-mx}$ prove that $\frac{d^2y}{dx^2} - m^2y = 0$

iii. If $y = \sin(\log x)$ prove that $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

iv. If $y = (\tan^{-1} x)^2$ prove that $(x^2+1)^2 \frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$

v. If $y = a\cos(\log x) + b\sin(\log x)$ then prove that $x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

vi. If $y = \log(x + \sqrt{x^2+1})$ prove that $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

60. If $x = \sin t, y = \sin pt$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$

61. If $x = \log t$ and $y = \frac{1}{t}$, show that $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

62. If $x = a\sin t - b\cos t,\ y = a\cos t + b\sin t$, show that $\frac{d^2y}{dx^2} = -\frac{x^2+y^2}{y^3}$

63. If $x = a(1-\cos\theta), y = a(\theta + \sin\theta)$ then find $\frac{d^2y}{dx^2}$ at $\theta = \pi/2$

64. If $x = a(1-\cos^3 t), y = a\sin^3 t$ find $\frac{d^2y}{dx^2}$ at $t = \pi/6$

65.Show that the equation of the perpendicular drawn to the hyperbola $xy = 16$ at the point $P \equiv \left(4t, \frac{4}{t}\right)$ in the first quadrant is $t^3x - ty - 4(t^4-1) = 0$. Where t is a parameter. If this perpendicular passes through the origin and meet the hyperbola at Q, find the coordinats of Q.

66.For the real parameter $t(\neq 0)$, show that the equation of tangent drawn to the curve C of equation $x^3 - y^2 = 0$ at a point $P(4t^2, 8t^3)$ is $3tx - y - 4t^3 = 0$. If the tangent drawn at P becomes perpendicular to the curve C at point $Q(4T^2, 8T^3)$, show that $T = -\frac{1}{9t}$

67.If the tangents drawn to the curve $x^2 = 4y$ at the points $P(2t, t^2)$ and $Q(4t, 4t^2)$ meet at the point R, show that the locus of the point R is given by $2x^2 = 9y$. Here, $t(\neq 0)$ is a parameter.

68.The ellipse C is parametrically given by $x = 3\cos\theta$, and $y = 2\sin\theta$. where $\theta \in [0, 2\pi]$.

Show that the equation of the perpendicular drawn to this ellipse C at $P \equiv (3\cos\theta, 2\sin\theta)$ where $\theta = \sin^{-1}\left(\frac{3}{5}\right)$ is $9x - 8y - 12 = 0$. If this perpendicular meets this ellipse at $Q \equiv \{3\cos\phi, 2\sin\phi\}$ Show that $27\cos\phi - 16\sin\phi = 12$.

69.A curve C is given by the parametric equations $x = ln\left(\tan\frac{t}{2}\right)$ and $y = \sin t$ for $\frac{\pi}{2} < t < \pi$.

Show that $\frac{dy}{dx} = \cos t \sin t$.

Deduce that the gradient of the tangent line drawn to the curve C at the point corresponding to $t = \frac{2\pi}{3}$ is $-\frac{\sqrt{3}}{4}$.

70. A balloon which always remains spherical is being inflated by pumping in $900\ cm^3$ of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is $150(cm)$

71. A particle moves along the curve $6y = x^3 + 2$. Find the points. On the curve at which the y-coordinate is changing 8 times as fast as the x- coordinate

72. A man $2\ m$, high walks at a uniform speed of $6\ kmh^{-1}$ away from a lamp post $6\ m$ high. Find the rate at which the length of his shadow increases.

73. A balloon gets inflated so that the rate of change of volume is proportional to its radius lnitially its radius is 2 units and after 1 second it is 3 units. Find its radius at time t.

74. Water is dripping out from a conical funnel of semi - vertical angle $\pi/4$ at the uniform rate $2\ cm^3/s$ through a tiny hole at the vertex at the bottom. When the slant height of the water is $4\ cm$, find the rate of decrease of the slant height of the water .

75.Two cities A, B are situated on the same side of a river. The shortest distance of A and B from nearest bank of the river are 4 (km) and 8(km) respectively.

The point on the nearest bank which is closest to A is D and that for B is C. Point C is such that $CD = 9(km)$.

A pumping station is to locate between C and D so that the length of the pipes required is minimum.

Find the position the pumping station should be setup.

76. A rectangle is inscribed in a semicircle of radius r with one of its sides on the diameter of the semi - circle. Find the dimensions of the rectangle so that its area is maximum. Find also this area.

77. ABC is a right angled variable triangle such that $B\hat{A}C = \frac{\pi}{2}$ and $BC = l$. where l is a constant.

Prove that when $AB = \frac{l}{\sqrt{2}}$

i) $AB^3 + AC^3$ is minimum

ii) $AB + AC$ is maximum

78. A cylinder of height 2h is cut from a sphere of radius 5 (cm). Show that the maximum volume of the cylinder is

$$\frac{500\sqrt{3}\ \pi}{9}.$$

79. Write down expressions for volume and total surface area of a closed cylinder by both ends of radius r and height h.

80.Derive an expression for the total area A of a cylinder of volume $300\pi\,(cm^3)$ in terms of base radius r.Show that the least area required to form above cylinder is $6\pi(150)^{\frac{2}{3}}$. Show also that the values of base radius r and heigh h related by $h=2r$.

81.A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is $8(m^3)$. If building of tank costs Rs:70 per square metre for the base and Rs:40 per square metre for sides. What is the cost of least expensive tank.

82.Let AP and BQ be two vertical poles at points A and B, respectively. If $AP=16\,m$, $BQ=22\ m$ and $AB=20\,m$, then find the distance of a point R on AB from the point A such that RP^2+RQ^2 is minimum.

83. If length of three sides of a trapezium other than base are equal to 10(cm). Find the maximum area of the trape zium. when the area is maximum find the length of the base.

84.A metal sheet of $40(m^2)$ are to be used in the construction of an open tank with a square base.Find the dimensions so that capacity is greatest possible.

85.Find the slpoe of the normal to the curve $x=a\cos^3\theta,\ y=a\sin^3\theta$ at $\theta=\frac{\pi}{4}$.

86.Find the equations of tangent and normal to the curve $y^2=\frac{x^2}{4-x}$ at $(3,-3)$.

87.Show that the equation of tangent to the ellispse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1,y_1) *is* $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

88.Find the point (s) on the curve $x^2+y^2-2x-4y+1=0$, where the tangent is

i. parallel to x - axis *ii*. parallel to y - axis.

89.Show that the curves $x=y^2$ and $xy=4$ intersect at right angles if $8k^2=1$.

90.Show that the normal at any point θ to the curve $x=a\cos\theta+a\theta.\sin\theta,\ y=a\sin\theta-a\theta.\cos\theta$ is at a constant distance from the origin.

91.The tangent and perpendicular drawn to the curve $y^2=x^3$ at (t^2,t^3) meet x- axis at P and Q.Show that $6PQ=9t^4+4t^2$. Find the coordinates of the point tangent meet x - axis.

92.Sketch the graph of each of the following.

i. $y=x^3-3x^2$ *ii.* $y=2x^3-3x^2-12x+13$

93. Sketch the graph of the function $y = x^4 - 4x^3 + 4x^2 + k$. Hence show that the equation $x^4 - 4x^3 + 4x^2 + k = 0$ has

*i.*Four roots when $-1 \le k \le 0$

*ii.*only two real roots when $k < -1$

*iii.*No real roots when $k > 0$.

94.Sketch the graph of $f(x) = \dfrac{x}{x^2 + 1}$. Hence sketch $y = |f(x)|$.

95.Sketch the graph of $y = \dfrac{x^3}{1 + x^4}$.