COMBINED MATHS
ANANDA ILLANGAKOON

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General Certificate of Education (Adv, Level) Examintaion, November 2024

සංයුක්ත ගණිතය **I**
Combined Mathematics **I**

10 **E** **I**

Three hours

∗ Answer all questions in part *A* and **five** questions only in part *B* . **Test paper** 04

PART A

1.Using the principle of mathematical induction, for all $n \in \mathbb{Z}^+$, prove that $\sum_{r=1}^{n}(4r+1) = n(2n+3)$.

2.Sketch in the same diagram, the paths of complex number z , satisfying

i.Arg $(z+1-3i) = -\frac{\pi}{4}$ and

ii.$|Z-2| = \sqrt{2}$

Hence find the complex number z represented by the point of intersection of these paths.

3.The parametric equation of a curve C is given by $x = 3\cos\theta$ and $y = 2\sin\theta$ for $0 < \theta < \frac{\pi}{2}$. Show that $\frac{dy}{dx} = -\frac{2\cot\theta}{3}$. If the gradient of the tangent drawn at point P on the curve C is $-\frac{2}{\sqrt{3}}$, find θ corresponding to point P. Also find the equation of the tangent at P.

4.If $y = \sin^{-1}x + (\sin^{-1}x)^2$, show that $(1-x^2)\frac{d^2y}{dx^2} - x.\frac{dy}{dx} - 2 = 0$

5.A line l with gradient m passes through the point $P(1,2)$. The perpendicular distance from the point $Q(2,3)$ to l is $\frac{1}{\sqrt{5}}$. Find the values of m.

6.Solve the equation $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$

7.Show that $\lim_{x \to \frac{\pi}{3}} \frac{\sin\left(x - \frac{\pi}{3}\right)}{\sqrt{3x} - \sqrt{\pi}} = \frac{2\sqrt{\pi}}{3}$.

8.Let $f(x) = x^4 + ax^3 + bx^2 - 17x + 6$ $(x+1)^2$ is a factor of $f(x)$. find a and b. Also find the other factor of $f(x)$.

9.Find the area enclosed by the curve $y = 9 - x^2$ and the line $x - y + 3 = 0$

see page 2

10.In the same diagram sketch the graphs of $y=3|x-1|$ and $y=|x|+3$. Hence or otherwise find all real values of x satisfying $3|2x-1|>2|x|+3$.

PART B

11.*a*.Let $f(x)=kx^2+(k-1)x+(1-2k)$, where $k\in\mathbb{R}^+$. Show that $f(x)=0$ has real roots. If the roots of $f(x)=0$ are α and β , find $\alpha+\beta$ and $\alpha\beta$ in terms of *k*. find the range of *k*, such that both roots are positive. Also find the equation with the roots α^2 and β^2. If a root of $f(x)=0$ is three times of the other root, find the values of *k*.

b.Let $f(x)=2x^3+ax^2+bx-9$. $(x-3)$ is a factor of *f*(*x*) and, when *f*(*x*) is divided by $(x+1)$ the remainder is 8. Find *a* , *b* and all factors of *f*(*x*). Hence solve the equation $16x^3+4ax^2=9-2bx$.

12. *a*.A committee of five members is to be formed from children of two schools *A* and *B* with boys and girls given below

school	Boys	Girls
A	3	4
B	7	5

For each of the following find the different number of committees.

i. Including any five.
ii. Representing both male and female.
iii. Any five including both schools *A* and *B*.

iv. Any five, representing both schools *A* and *B* and it is compulsory to represent male and female from each school.

b.Let $\lambda>0\ (\neq 1)$ and $r\in\mathbb{Z}^+$.

Show that $\dfrac{2}{r+\lambda}-\dfrac{2}{r+\lambda-2}=\dfrac{-4}{(r+\lambda)(r+\lambda-2)}$.

Hence find, v_r such that $u_r=v_r-v_{r+2}$. Where $u_r=\dfrac{2}{(r+\lambda)(r+\lambda-2)}$.

Prove that $\displaystyle\sum_{r=1}^{n}u_r=\frac{2\lambda-1}{\lambda(\lambda-1)}-\left[\frac{2(\lambda+n)-1}{(n+\lambda)(n+\lambda-1)}\right]$.

Show that the series $\displaystyle\sum_{r=1}^{\infty}u_r$ is finite and find its sum. By giving a suitable value for λ deduce that

$$\sum_{r=1}^{\infty}\frac{2}{(r+1)(r+3)}=\frac{5}{6}.$$

13.*a*.The matrices $P=\begin{bmatrix}1 & 0\\ 0 & \lambda\\ \lambda & -2\end{bmatrix}$, $Q=\begin{bmatrix}-2 & \lambda\\ 3 & 4\\ 0 & -1\end{bmatrix}$ and $R=\begin{bmatrix}\mu-1 & 0\\ -3 & \mu-1\end{bmatrix}$ are three matrices such that $P^TQ=R$.

Where $\lambda,\mu\in\mathbb{R}$. Show that $\lambda=\mu=-1$. Write down matrix *R*. Now by considering this matrix *R* and the matrix

$A = \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{3}{4} & -\frac{1}{2} \end{pmatrix}$, show that $A = R^{-1}$. When $S = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$

Show that *i.* $(R+I).\ S = -S$ and

ii. $R + 2I + S = 0$

Hence deduce that $(R+2I)\ (S-I) = S$. Where *I* is the unit matrix of order 2.

*b.*Let $z_1 = -1 + 2i$ and $z_2 = 2 + i$

Find $\frac{z_1}{z_2}$ and deduce $\frac{z_2}{z_1}$. Hence obtaining $\frac{z_1 + z_2}{z_1}$ and $\frac{z_1 + z_2}{z_2}$

deduce that *i.* $\frac{z_1 + z_2}{z_2} + \frac{z_1 + z_2}{z_1} = 2$ and

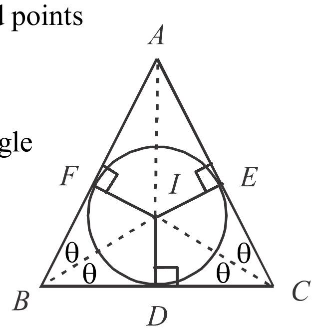
ii. $\frac{(z_1)^2 - (z_2)^2}{z_1 z_2} = 2i$.

14.*a.*Let, $f(x) = \frac{x(x+3)}{(x+1)^2}$, for $x \neq -1$. Show that $f'(x)$ the first derivative of *f*(*x*) is given by $f'(x) = \frac{3-x}{(x+1)^3}$.

Hence find the range of *f*(*x*) increasing and *f*(*x*) decreasing. Also show that the second derivative $f''(x)$ of *f*(*x*) is given by $f''(x) = \frac{2(x-5)}{(x+1)^4}$.

Sketch the graph of $y = f(x)$ indicating, asymptotes, *y* - intercepts, turning points, and points of inflection.

*b.*In the triangle *ABC*, $AB = AC$ and $A\hat{C}B = 2\theta$. The radius of the in - circle of the triangle *ABC* is *a* and its centre is *I*. Where *a* is a constant. If the perimeter of *ABC* triangle is *P*. Show that $P = 4a\cot\theta + 2a\tan 2\theta$. Show that the minimum value of *P* is $6\sqrt{3a}$.



15.*a.*For all $x \in \mathbb{R}$, it is given that $x^3 + 5x^2 + 14x + 29 = A(x+2)(x^2+9) + (2x+B)\ (x+2) + (x^2+9)$.

Find the values of *A* and *B* .

Hence write $\frac{x^3 + 5x^2 + 14x + 29}{(x+2)(x^2+9)}$ in partial fractions and find $\int \frac{x^3 + 5x^2 + 14x + 29}{(x+2)(x^2+9)} . dx$.

*b.*Using integration by parts evaluate $\int_0^{\frac{\pi}{3}} 2\sin x . \ln(\sec x).\ dx$

*c.*for $0 \le \theta \le \frac{\pi}{4}$ using the substitution $x = 2\cos 2\theta$, show that $\int_0^2 \sqrt{\frac{2-x}{2+x}} . dx = \pi - 2$.

Hence deduce the value of $\int_0^{\sqrt{2}} x\sqrt{\frac{2-x^2}{2+x^2}} . dx$

16. Find the coordinates of the point of intersection P of the lines $l_1 \equiv y = mx$ and $l_2 \equiv 2mx - 3y + 1 = 0$, where $m > 0$. If this point P is situated a distance $\sqrt{2m}$ from the origin. show that $m = 1$.

Find the equation of the line l_3 that passes through the point of intersection of $l_1 = 0$ and $l_2 = 0$ making an intercept of 2(units) on positive x - axis .

Let the line $l_2 = 0$ cuts y - axis at A and $l_3 = 0$ cuts x - axis at B. Find the equation $S_1 = 0$ of the circle passing through the points O, A and B.

Also find the equation $S_2 = 0$ of the circle with P as the centre and radius PA. Do the circles $S_1 = 0$ and $S_2 = 0$ intersect orthogonally.? justify your answer. find the equation of the circle with P as the centre and orthogonal to $S_1 = 0$.

17.*a*.Let $f(x) = 11\cos^2 x + 16\cos x.\sin x - \sin^2 x$. Express $f(x)$ in the form $a + b\cos(2x - \alpha)$. where a, b and α are constants to be determined.

Hence for $0 \le x \le \pi$, sketch the graph of $y = f(x)$.

Find the solutions of $f(x) = 0$ in $\theta \le x \le \pi$.

b.For any triangle *ABC*, state and prove “cosine” rule.

In the triangle *ABC*, the lengths of the sides *BC, CA, AB* are $a, a + d$, $a + 2d$, respectively.

Prove that $\cos C = \frac{1}{2} - \frac{3d}{2a}$.

Hence find the range of values of $\frac{d}{a}$ such that $\frac{2\pi}{3} < C < \pi$.

c.Solve the equation $\tan^{-1}(5\tan^2 x) + \tan^{-1}(\cos^2 x) = \frac{\pi}{4}$.

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