Numbers and Functions

Real Numbers

All the numbers available on the real number line are called as **real** numbers.

i. positive and negative whole numbers. *ii.* positive and negative proper fractions

iii. positive and negative improper fractions *iv.* positive and negative finite decimal

v. positive and negative infinite decimal *vi.* positive and negative recurring decimal

vii. zero

The set containing all real numbers is called as set of real numbers. ($\mathbb{R}$).

$\mathbb{R}^+ \equiv$ set of positive real numbers $\mathbb{R}_0^+ \equiv$ set of positive real including zero

$\mathbb{R}^- \equiv$ set of negative real numbers. $\mathbb{R}_0^- \equiv$ set of negative real numbers including zero.

Natural numbers

Positive whole numbers are called as natural numbers. Set of natural numbers is denoted by $\mathbb{N}$.

$\mathbb{N} = \{1, 2, 3, 4, ...\}$

Integers

All positive and negative whole numbers with zero are called as integers. Set of integers is denoted by $\mathbb{Z}$

$\mathbb{Z} = \{......., -3, -2, -1, 0, 1, 2, 3, ...\}$.

$\mathbb{Z}^+ =$ set of all positive intergers . $\mathbb{Z}^+ = \{1, 2, 3, 4, ...\}$

$\mathbb{Z}^- =$ set of all negative integers . $\mathbb{Z}^- = \{........., -3, -2, -1\}$

$\mathbb{Z}_0^+ =$ set of all positive integers with zero. $\mathbb{Z}_0^+ = \{0, 1, 2, 3.......\}$

$\mathbb{Z}_0^- =$ set of all nagative integers with zero. $\mathbb{Z}_0^- = \{........., -3, -2, -1, 0\}$

Rational numbers

Let x be a rational number. Then $x = \frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{N},\ (p, q) = 1$ $\left(\begin{matrix}\text{Highest common factor} \\ \text{of } p \text{ and } q \text{ is } 1\end{matrix}\right)$

Any simplest fraction is a rational number

$\frac{1}{2}, 4, -\frac{3}{2}, \frac{1}{3}, \frac{1}{4}, ...$

A rational number can be a

i. Integer (4) ii. Finite decimal $\left(0.25 = \frac{1}{4}\right)$ iii. Recurring decimal $\left(\frac{1}{3} = 0.333......\right)$

Set of rational numbers is denoted by $\mathbb{Q}$

$$\mathbb{Q} = \left\{ x / x = \frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{N},\ (p,q) = 1 \right\}$$

Irrational numbers.

The numbers which cannot be written in the form $\frac{p}{q}$, $p \in \mathbb{Z}, q \in \mathbb{N}$ are called as Irrational numbers.

$\sqrt{2}, \sqrt{3}, \sqrt{13}, \pi,.....$ These numbers cannot be expressed as fractions.

Decimal representation of an irrational number is an infinite decimal. (No patten of decimal representation).

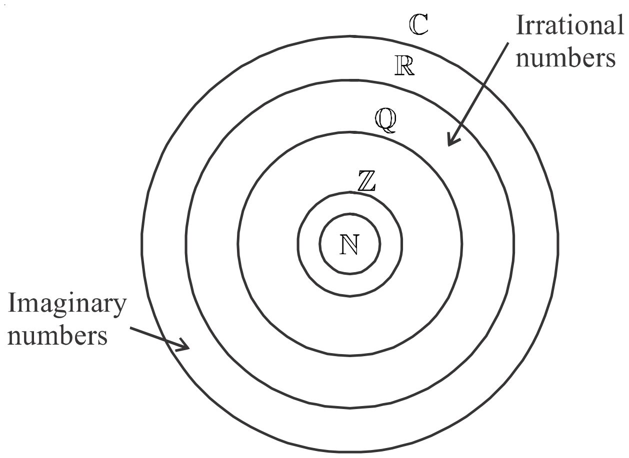
Complex numbers

$z = x + iy,\ x, y \in \mathbb{R},\ i = \sqrt{-1} \equiv$ imaginary unit.

$2 + 3i$, $1 - 5i, -3 + 4i, 5, 10i....$

$*$ All real numbers are complex numbers.

$*$ iy is called imaginary number $2i, -3i,.....$



FUNCTIONS

RELATIONS

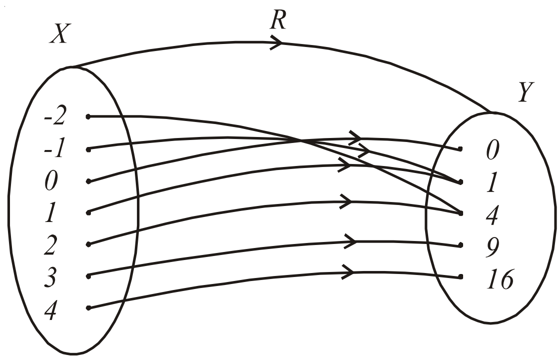
If there exists a correspondence between the elements of two given sets, then it is said that there is a relation between the given sets.

For instance

consider the sets $X = \{-1, -1, 0, 1, 2, 3, 4\}$

$Y = \{0, 1, 4, 9, 16\}$

The elements in the set X can be jointed to the elements in the set Y under the rule "squaring" as follows



R **= Squaring**

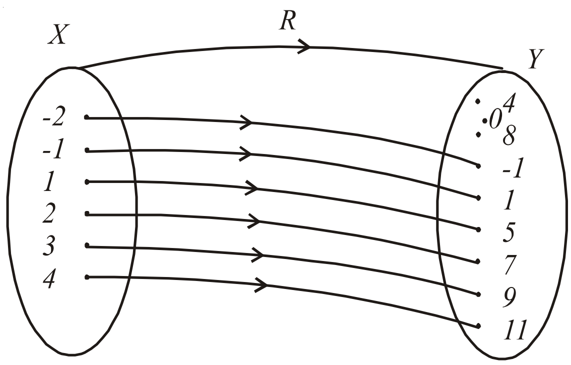
Therefore a relation means, it is a statement or a rule which joins elements in two sets.

The method of expressing the relation "R" existing between the two sets X and Y is written as

$$R : X \rightarrow Y$$

<u>Domain, Co-domain, Image, Pre-image and range of a relaltion</u>

Consider the following relation from the set X to Y.



R ≡ multiply by 2 and add 3

Domain

* The set from which the relation starts is defined as the "Domain" of the relation. Thus the domain of the above relation R is X, where $X = \{-2, -1, 1, 2, 3, 4\}$.

Co-domain

* The set at which the relation terminates is defined as the "Co-domain" of the relation. Thus the co-domain of the above relation R is Y, where $Y = \{4, 0, 8, -1, 1, 5, 7, 9, 11\}$.

Image

* The elements in the co-domain to which arrows are coming from the domain are defined as images.

Pre-image

The elements in the domain from which arrows are going to co-domain are defined as pre-images of the images.

Range

* The set containing all images is defined as the range of the relation.

  the range of the above relation is $\{-1,1,5,7,9,11\}$

* The range is always a subset of the co-domain
* A relation is known as a mapping.
* The above arrow diagram is known as a mapping diagram.

Functions

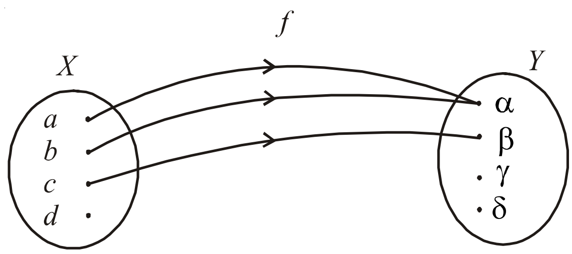
A relation is defined as function if the relation satisfies the properties mentioned below.

1. Every element in the domain has an image in the co-domain.
2. The image of any element in the domain must be unique.

   (there sould be only one image for any element in the domain)

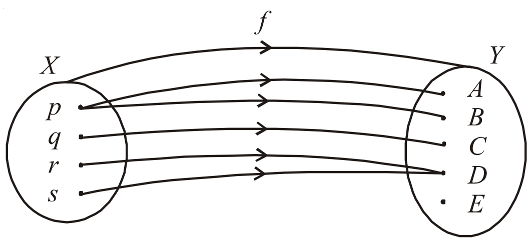
Eg :-

1.



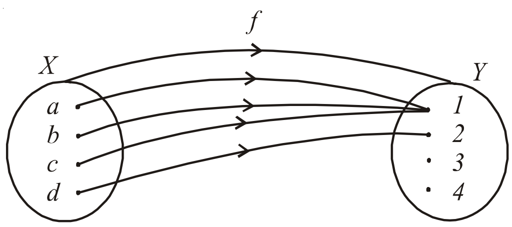
f is not a function, because the element *d* in *X* (domain) doesnot have an image

2.



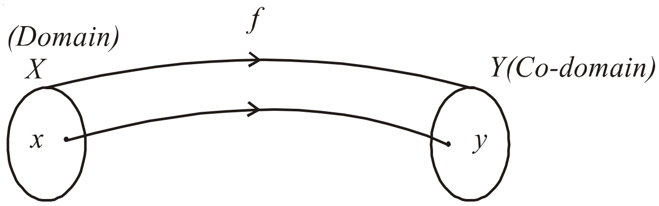
f is not a function, because the element p in the domain *X* has two images.

3.



f is afunction

Note : Suppose there is a function *"f"* from set *X* to *Y*.



i. This function is written as $f: X \rightarrow Y$

ii. *y* is the image of *x* under *f*. It is written as y = *f(x)*

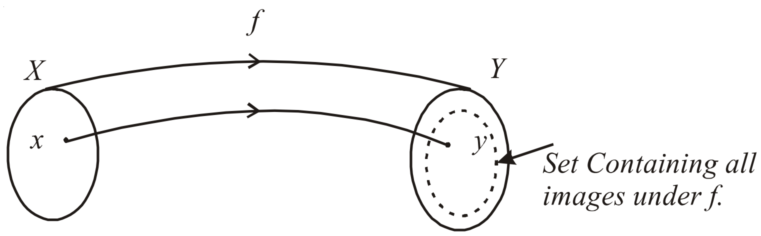
iii. *f(x)* is the image of *x* under *f* and *f(x)* is therefore in the co-domain.

iv. *f(x)* means that after applying the rule given by *"f"* on the element *x* , the output result.

Here , *x* is called as independent variable.

y is called as dependent variable.

Note :



$X \equiv$ Domain
$Y \equiv$ Co-domain

$* \quad f: X \rightarrow Y$

$y = f(x)$

Real Functions (Real valued functions)

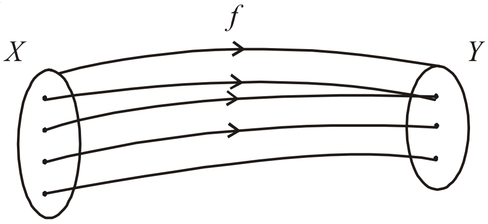
If the domain and co-domain of a function *'f"* are subsets of set of real numbers, then the funciton *'f"* is called a real function.

Types of Functions

01. Many - one function

A function $f: X \rightarrow Y$ is said to be many - one if there exist at least two distinct elements in X (domain) whose images are same.

Eg 1).



Eg2). $f: \Re \rightarrow \Re$

$f(x) = x^2$

f is many - one function.

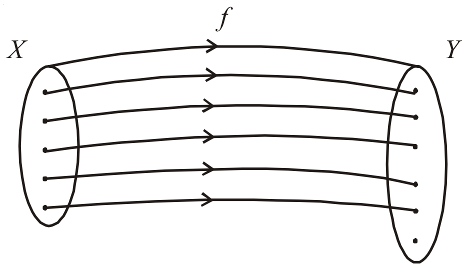
Eg 3). $f: \{-1,1,2,3\} \rightarrow \{1,4,9\}$

$f(x) = x^2$ is many - one function.

02. *One-one function (injective function)*

A function $f: X \rightarrow Y$ is said to be one-one or injective, if every image in Y (co-domain) has only one pre- image in X (domain)

Eg 1).



Eg 2). $f : \Re \rightarrow \Re$

$f(x) = 2x$

f is one - one function

Eg 3). $f : \{1,2,3\} \rightarrow \{3,5,7,10\}$

$f(x) = 2x + 1$

f is one - one function

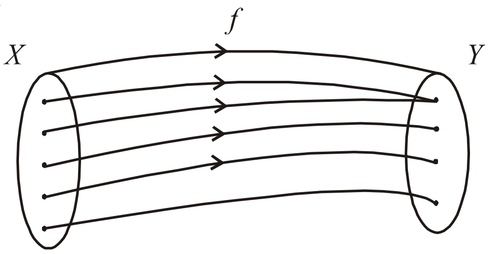
Note :

1. If f is is one-one function, its number of images and number of pre-images are same.
2. In order to show a function f is one-one take any two images $f(x_1)$ and $f(x_2)$ from co-domain, and assuming that $f(x_1) = f(x_2)$ and hence show that $x_1 = x_2$.

03. *Onto Function (Surjective Function)*

A function $f: X \rightarrow Y$ is said to be onto or surjective function, if every element in co-domain is an image.

Eg 1).



f is an on to function

Eg 2). $f : \Re \rightarrow \Re$ Eg

$f(x) = 2x + 6$

f is an on to function

Eg 3). $f : \{-2,2,4,6\} \rightarrow \{4,16,36\}$

$f(x) = x^2$

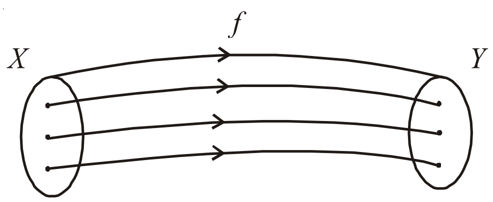
f is an on to function

Note :

In order to show a function is an onto function, we have to show for every element in co-domain, there exists a pre-image in domain.

04. *Bijective Function*

A function $f: X \rightarrow Y$ is said to be bijective if f is both one - one and onto.



Eg 1). $f : \Re \rightarrow \Re$

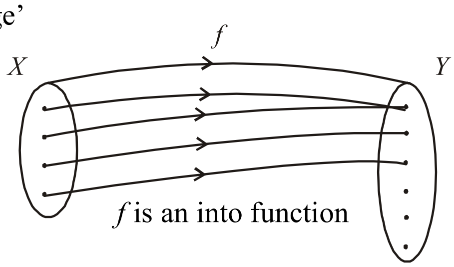
$f(x) = x + 1$

f is a bijective function.

05. ***Into Function***

A function $f: X \rightarrow Y$ is said to be an into function if there is at least one element in co-domain such that it is not an image'

Eg 1).



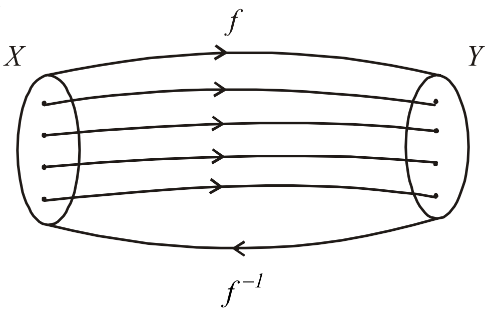
Eg 2) $f: \Re_0^+ \rightarrow \Re$

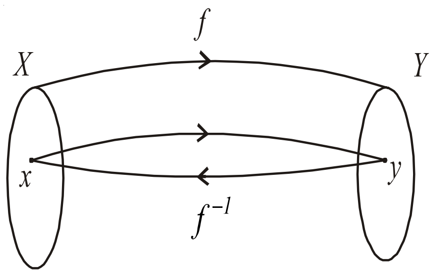
$f(x) = \sqrt{x}$ (positive squrare root)

f is an into function

06. ***Inverse Function***

If a function $f: X \rightarrow Y$ is a bijective function (one-one and onto) then it is possible to find a rule (function) from Y to X. This function is defined as the inverse function of f and is denoted by f^{-1}.





$f: X \rightarrow Y$ $\qquad$ $f^{-1}: Y \rightarrow X$

$f(x) = y$ $\qquad$ $f^{-1}(y) = x$

Graph of a function

Consider the function

$f: X \rightarrow Y$

$y = f(x)$

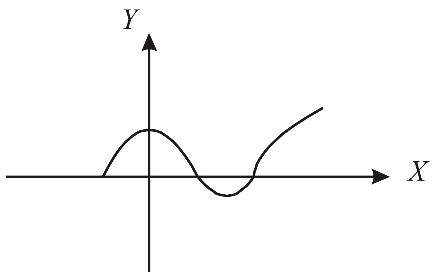
We can form a set G such that $G = \{(x, f(x)) / x \in X(domain), f(x) \in Y(co-domain)\}$

the set G is defined as the graph of the function f.

Geometric Presentaion of Graph of a Function

The elements of the form *(x, f(x))* of the set G are represented by points on the plane of the axes X and Y. Then the set G can be shown by collection of some points.

This collection of points on *OXY* plane looks like as a curve.



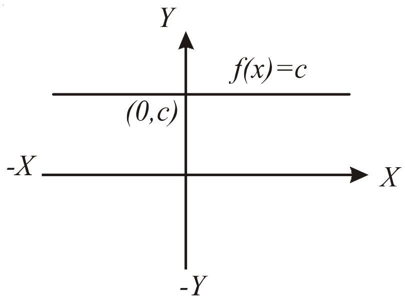
Some Important Functions

01. *Constant Function*

A function $f:\Re \to \Re$ defined by $f(x)=c$, for all $x \in \Re$ where c is a constant is called a constant function.

Its domain is $\Re$ and range is the set $\{c\}$

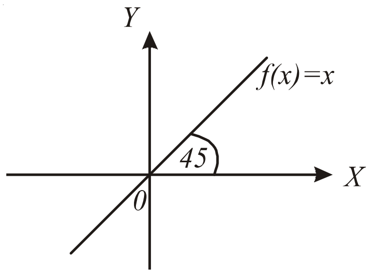
The graph of a constant function is a straight line parallel to x-axis when x is the independent variable.



02. *Identity Function (Unit Function)*

The function $f:\Re \to \Re$ defined by $f(x)=x$ for all $x \in \Re$, is called the identity function. Its domain is $\Re$ and range is also $\Re$.

The graph of the identity function is a straight line passing through origin and inclined at an angle of 45^0 with positive $x-$axis.



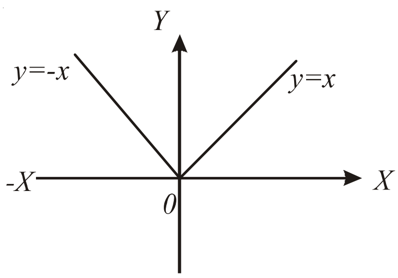
03. *Modulus Function (Absolute Value Function)*

The function $f:\Re \to \Re$ defined by

$$f(x)=|x|=\begin{cases} x; \text{ if } & x \geq 0 \\ -x; \text{ if } & x < 0 \end{cases}$$

is called the absolute value function. Its domain is $\Re$ and its range is $[0,\infty)$.

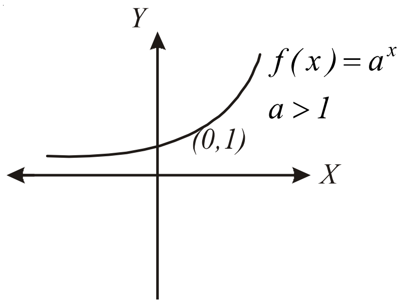
The graph of the modules function is as shown below.

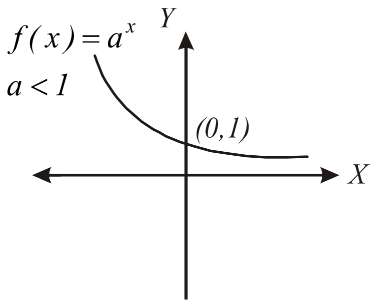


04. *Exponential Function*

Let $a(\neq 1)$ be a positive real number.

Then the function $f:\Re \to \Re$, defined by $f(x)=a^x$ is called the exponential function. Its domain is $\Re$ and range $(0,\infty)$. The graph of the exponential function is as shown below





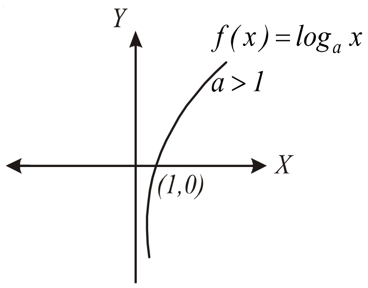
05. *Logarithmic Function*

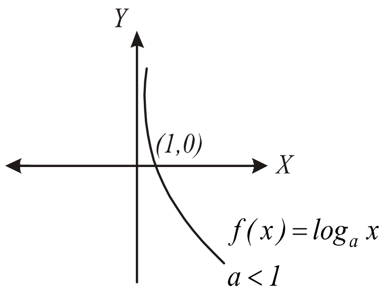
Let $a(\neq 1)$ be a positive real number.

Then the function $f:(0,\infty) \rightarrow R$ defined by

$f(x) = \log_a x$ is called the logarithmic function.

Its domain is $(0,\infty)$ and range is R. The graph of logarithmic function is as shown below.





06. *Polynomial Function*

A function $f: \Re \rightarrow \Re$ defined by

$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ where n is a positive integer and $a_0, a_1, a_2, ...a_n$ are real numbers is called a polynomial function.

If $a_n \neq 0$ then n is called the degree of the polynomial. The domain of the polynomial function is R.

07.*Rational Function*

A function of the form $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials over the set of real numbers and $q(x) \neq 0$ is called a rational function. Its domain is $\Re \setminus \{x / q(x) \neq 0\}$.

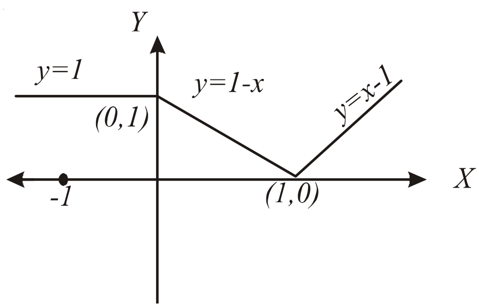
08. *Piecewise function*

Definition

If a function $y = f(x), x \in [a,b]$ assumes different forms in different subsets of $[a,b]$ we say that it is piecewise defined

For instance $y = f(x) = \begin{cases} 1; & -1 \leq x < 0 \\ 1 - x; & 0 \leq x < 1 \\ x - 1; & x \geq 1 \end{cases}$

The graph is



09. *Explicit and Implicit Functions*

Explict Function

A function y is said to be an explicit function of x, if the dependent variable y can be expressed totally in terms of the independent variable x.

Illustration

1. $y = x + log\ x$ 2. $y = tan\ x + x$ 3. $y = x^2 + 2x - 1$

Implicit Function

When the variable x and y occur together in an equation, in which y cannot be expressed explicitly in terms of x, then y is said to be an implicit function of x.

1. $x^2 y = (1 - x)(y + 1)$ 2. $e^{xy} + xy - 2 = 0$

Odd and even functions

Odd function

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$ for all x.

$y = f(x) = sin\, x$, $f(x) = x^3$ are odd functions.

Even functions

A function $f(x)$ is said to be even if $f(-x) = f(x)$ for all x.

$y = f(x) = x^2$ is an even function.

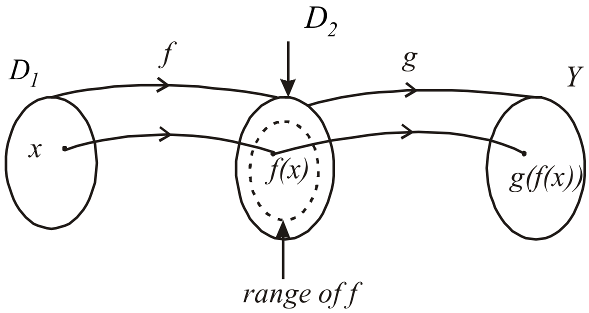
Composition of Functions

Let f and g be real functions with domain D_1 and D_2 respectively.

If range of f is a subset of domain of g.

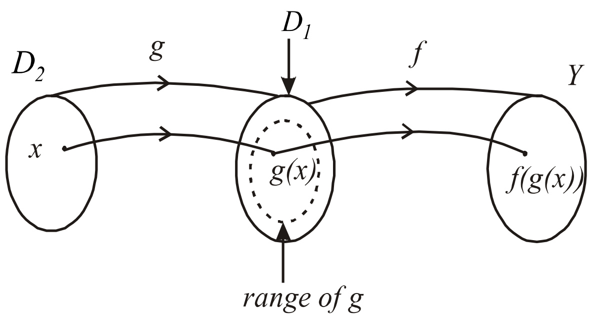
Then gof (composition of f and g) is defined as

$(gof)(x) = g(f(x))$ for all $x \in D_1$



Similarly if range of g is a subset of domain of f.
Then fog (composition of g and f) is defined as,

$fog(x) = f(g(x))$ for all $x \in D_2$



Operations on Functions

Let f and g be real functions with domain D_1 and D_2 respectively.
Then

i. The sum function *(f+g)* is defined by

$(f+g)(x) = f(x) + g(x)$ for all $x \in D_1 \cap D_2$ The domain of $f+g$ is $D_1 \cap D_2$

ii. The difference function *(f-g)* is defined by

$(f-g)(x) = f(x) - g(x)$, for all $x \in D_1 \cap D_2$ domain of $f-g$ is $D_1 \cap D_2$.

iii. The product function *fg* is defined by

$(fg)(x) = f(x).g(x)$, for all $x \in D_1 \cap D_2$ domain of fg is $D_1 \cap D_2$

iv. The quotient function $\left(f/g\right)$ is defined by

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, for all $x \in D_1 \cap D_2 \setminus \{x : g(x) = 0\}$.The domain of $\frac{f}{g}$ is $D_1 \cap D_2 \setminus \{x : g(x) = 0\}$

v. The scalar multiple function *cf* is defined by

$(cf)x = c.f(x)$, for all $x \in D_1$. The domain of Cf is D_1.

Exercise

1*a*.Find the domain and range of the following functions.

i. $f(x) = \sqrt{x^3 + 1}$

ii. $f(x) = \sqrt{(x-2)(x+3)}$

b.Two functions *f* and *g* are defined as follows.

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x - 1 \; ; x < 1 \\ 1 \quad ; 1 \le x < 3 \\ 1 - x \; ; x \ge 3 \end{cases}$$

$g : \mathbb{R} \to \mathbb{R}$

$$g(x) = \begin{cases} x^2 + 1 \; ; x < 2 \\ x + 1 \quad ; x \ge 2 \end{cases}$$

Express the function $f(x) + g(x)$ as in the above form.

c.when $x \in \mathbb{R}_0^+$ let $f(x) = x^2$. Find the function $f^{-1}(x)$. Sketch the graph of $f(x) = x^2$ and hence sketch the graph of $f^{-1}(x)$.

2.Find the implied domain and range of the following real valued functions.

i. $f(x) = \sqrt{4x^2 - 100}$ *ii*. $f(x) = 3 + x - \frac{x^2}{x-1}$

3.Sketch the graph of the function $f(x)$ given by

$$f(x) = \begin{cases} x^2 - 1 \; ; x < 0 \\ -3 \quad ; 0 \le x < 3 \\ 2x + 1 \; ; x \ge 3 \end{cases}$$

4.Find the domain and range of the following functions.

i. $f(x) = x - 2$ ii. $g(x) = -2x$ iii. $f(x) = x^2 - 4$ iv. $g(x) = -x^2 + 4$ v. $f(x) = x^2 - 2$

vi. $f(x) = -\frac{1}{2}x^2 + 2x$ vii. $f(x) = x^3$ viii. $f(x) = \sqrt{x}$ ix. $f(x) = \sqrt{x-2}$ x. $f(x) = \sqrt{x+2}$

xi. $f(x) = \sqrt{x-2} + 1$ xii. $f(x) = \sin x$ xiii. $f(x) = -2\sin x$ xiv. $f(x) = \cos x$ xv. $f(x) = \cos^2 x$

xvi. $f(x)=e^{x}$ xvii. $f(x)=e^{x}-2$ xviii. $f(x)=e^{-x}$ xix. $f(x)=e^{-x}+2$ xx. $f(x)=2e^{x}$

xxi. $f(x)=\dfrac{1}{e^{x}+1}$ xxii. $f(x)=|x|$ xxiii. $f(x)=|x-2|$ xxiv. $f(x)=|x|-2$ xxv. $f(x)=|x-3|-1$

5.*i*. Let $x=\dfrac{1-y^2}{1+y^2}$. Using this, form a function as $y=f(x)$, indicationg domain and range.

ii. A function *f* is defined as

$$f:\mathbb{R}\rightarrow\mathbb{R}$$

$$f(x)=\frac{x}{2x+1}$$

Find the implied (natural) domain and range of this function.

Show also that the function *f* is a bijective function. Find f^{-1}.

6.Find the domain and range of the function defined by

$$f(x)=\frac{x^2-3x+2}{x^2+x-6}$$

7.Let $f(x)=5-3x^2$, and $g(x)=\sqrt{x-2}$

Then find fog.

8.Let $f(x)=\sqrt{x+2}+\dfrac{1}{\sqrt{3-x}}$

Find the domain of the function *f*.

9.For each of the following functions *f*(*x*) and *g*(*x*) find

(1) fog (2) gof, (3) fof (4) gog

1. $f(x)=2x+3$ $g(x)=3x$ ii. $f(x)=3x+1$ $g(x)=x^2$ iii. $f(x)=x^2$, $g(x)=x^2+4$

iv. $f(x)=\dfrac{x}{x-1}$, $g(x)=-\dfrac{4}{x}$ v. $f(x)=\sqrt{x}$, $g(x)=2x+3$ vi. $f(x)=x^2+1$, $g(x)=\sqrt{x-1}$

vii. $f(x)=\dfrac{x-5}{x+1}$, $g(x)=\dfrac{x+2}{x-3}$ viii. $f(x)=-x$, $g(x)=2x-4$ ix. $f(x)=x+1$, $g(x)=x^2+4$

x. $f(x)=\dfrac{1}{x+3}$, $g(x)=-\dfrac{2}{x}$ xi. $f(x)=\dfrac{x}{x+1}$, $g(x)=\dfrac{1}{x^2-4}$

10. Let $f(x)=2x$ and $g(x)=x-1$

Find

a. $(f+g)(x)$ *b*. $(f-g)(x)$ *c*. $(fg)(x)$ *d*. $\left(\dfrac{f}{g}\right)(x)$ *e*. $(kf)(x)$, where *k* is a constant.